Math E-21b – Spring 2025 – Homework #13

The Main Idea:

Given a system of 1st order linear differential equations $\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$ with initial conditions $\mathbf{x}(0)$, we use eigenvalue-eigenvector analysis to find an appropriate basis $\mathcal{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ for $\mathbf{R}^n$ and a change of basis matrix $\mathbf{S} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix}$ such that in coordinates relative to this basis ($\mathbf{u} = \mathbf{S}^{-1}\mathbf{x}$) the system is in a standard form with a known solution. Specifically, we find a standard matrix $\mathbf{B} = [\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{AS}$, transform the system into $\frac{d\mathbf{u}}{dt} = \mathbf{Bu}$, solve it as $\mathbf{u}(t) = [e^{t\mathbf{B}}]\mathbf{u}(0)$ where $[e^{t\mathbf{B}}]$ is the ***evolution matrix*** for **B**, then transform back to the original coordinates to get $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$ where $[e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$ is the ***evolution matrix*** for **B**. That is $\mathbf{x}(t) = [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)$. There are just three main cases to consider: (a) the (real) diagonalizable case, (b) complex eigenvalues, and (c) repeated eigenvalues with geometric multiplicity strictly less than the algebraic multiplicity.

There is also an **alternative formulation**, especially in the diagonalizable case, that is less reliant on matrices. Specifically, if the eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$ yield a basis of corresponding eigenvectors $\mathcal{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, then all solutions will be of the form $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n$ where the coefficients $c_1, \cdots, c_n$ are determined from the initial conditions, i.e. $\mathbf{x}(0) = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$.

Problem 1. Consider a linear system $\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$ of arbitrary size. Suppose $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions of this system. Show that the sum $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ is also a solution for arbitrary scalars c_1 and c_2.

In the following exercises, (a) solve the system with the given initial value and (b) sketch rough phase portraits for the dynamical systems (or use the Java tool to sketch the underlying vector fields and some trajectories).

Problem 2. $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

Problem 3. $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 4 & 3 \\ 4 & 8 \end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 4. $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

Problem 5. $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

[Sketching the 3D phase portrait is optional for this one.]

Problem 6. Solve the system $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
Give the solution in real form. Sketch the solution.

Problem 7. Solve the system $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 7 & 10 \\ -4 & -5 \end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
Give the solution in real form. Sketch the solution.

Problem 8. Consider the system $\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$ with $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. (a) Sketch a direction field for **Ax** (or use the Java tool) and, based on your sketch, describe the trajectories geometrically. (b) Find the solutions analytically, i.e. give a formula for the general solution.

Problem 9. Let **A** be an $n \times n$ matrix and k a scalar.

Consider the following two systems: $$\left\{ \begin{array}{ll} \dfrac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} & \text{(I)} \\ \dfrac{d\mathbf{c}}{dt} = (\mathbf{A} + k\,\mathbf{I}_n)\mathbf{c} & \text{(II)} \end{array} \right\}$$

Show that if $\mathbf{x}(t)$ is a solution of system (I), then $\mathbf{c}(t) = e^{kt}\mathbf{x}(t)$ is a solution of system (II).

Problem 10. Find all solutions of the system $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\mathbf{x}$, where λ is an arbitrary constant. *Hint*: Problems 8 and 9 are helpful. Sketch a phase portrait. For which choices of λ is the zero state a stable equilibrium solution?

Problem 11. Find the general solution to the system: $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}\mathbf{x}$.

Compare this with Problem 9. When is the zero state a stable equilibrium solution?

Problem 12. Consider the interaction of two species of animals in a habitat. We are told that the change of the populations $\mathbf{x}(t)$ and $\mathbf{y}(t)$ can be modeled by the equations

$$\left\{ \begin{array}{l} \dfrac{dx}{dt} = 1.4x - 1.2y \\ \dfrac{dy}{dt} = 0.8x - 1.4y \end{array} \right\}$$ where time t is measured in years.

a. What kind of interaction do we observe (symbiosis, competition, or predator-prey)?

b. Sketch a phase portrait for this system. From the nature of the problem, we are interested only in the first quadrant.

c. What will happen in the long term? Does the outcome depend on the initial populations? If so, how?

Problem 13. For each of the linear systems in A through E, find the matching phase portrait (to the right). Briefly justify your choices by examining the eigenvalues.

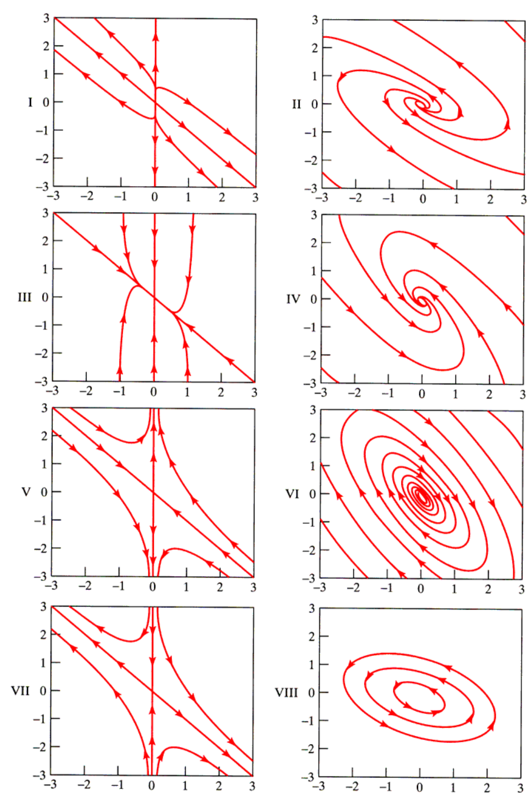
A. $\mathbf{x}(t+1) = \begin{bmatrix} 3 & 0 \\ -2.5 & 0.5 \end{bmatrix}\mathbf{x}(t)$

B. $\mathbf{x}(t+1) = \begin{bmatrix} -1.5 & -1 \\ 2 & 0.5 \end{bmatrix}\mathbf{x}(t)$

C. $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & 0 \\ -2.5 & 0.5 \end{bmatrix}\mathbf{x}$

D. $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} -1.5 & -1 \\ 2 & 0.5 \end{bmatrix}\mathbf{x}$

E. $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}\mathbf{x}$



For additional practice:
Section 9.1:
Solve the following initial value problems and graph the solution:

4. $\dfrac{dy}{dt} = 0.8t$ with $y(0) = -0.8$

5. $\dfrac{dy}{dt} = 0.8y$ with $y(0) = -0.8$

43. Consider the interaction of two species of animals in a habitat. We are told that the change of the populations **x**(t) and **y**(t) can be modeled by the equations

$$\begin{cases} \dfrac{dx}{dt} = 5x - y \\ \dfrac{dy}{dt} = -2x + 4y \end{cases}$$

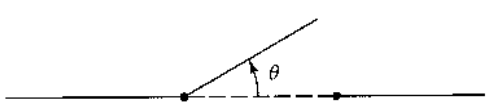
where time t is measured in years.
a. What kind of interaction do we observe (symbiosis, competition, or predator-prey)?
b. Sketch a phase portrait for this system. From the nature of the problem, we are interested only in the first quadrant.
c. What will happen in the long term? Does the outcome depend on the initial populations? If so, how?

49. Here is a continuous model of a person's glucose regulatory system. (Compare this with Exercise 7.1.52.) Let $g(t)$ and $h(t)$ be the excess glucose and insulin concentrations in a person's blood. We are told that

$$\begin{cases} \dfrac{dg}{dt} = -g - 0.2h \\ \dfrac{dh}{dt} = 0.6g - 0.2h \end{cases}$$

where time t is measured in hours. After a heavy holiday dinner, we measure $g(0) = 30$ and $h(0) = 0$. Find closed formulas for $g(t)$ and $h(t)$. Sketch the trajectory.

54. Consider a door that opens to only one side (as most doors do). A spring mechanism closes the door automatically. The state of the door at any given time t (measured in seconds) is determined by the angular displacement $\theta(t)$ (measured in radians) and the angular velocity $\omega(t) = \dfrac{d\theta}{dt}$. Note that θ is always positive or zero (since the door opens to only one side), but ω can be positive or negative (depending on whether the door is opening or closing).



When the door is moving freely (nobody is pushing or pulling), its movement is subject to the following differential equations:

$$\begin{cases} \dfrac{d\theta}{dt} = \omega & \text{(the definition of } \omega\text{)} \\ \dfrac{d\omega}{dt} = -2\theta - 3\omega & \text{(}-2\theta\text{ reflects the force of the spring, and }-3\omega\text{ models friction)} \end{cases}$$

a. Sketch the phase portrait of this system.
b. Discuss the movement of the door represented by the qualitatively different trajectories. For which initial states does the door slam (i,e,, reach $\theta = 0$ with velocity $\omega < 0$)?

55. Answer the questions posed in Exercise 54 for the system $\begin{cases} \dfrac{d\theta}{dt} = \omega \\ \dfrac{d\omega}{dt} = -p\theta - q\omega \end{cases}$

where p and q are positive, and $q^2 > 4p$.