Math E-21b – Spring 2019 – Homework #13

Section 9.1:

24. Let $\mathbf{A}$ be an $n \times n$ matrix and k a scalar. Consider the following two systems: $\left\{ \begin{matrix} \dfrac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} & \text{(I)} \\ \dfrac{d\mathbf{c}}{dt} = (\mathbf{A} + k\,\mathbf{I}_n)\mathbf{c} & \text{(II)} \end{matrix} \right\}$

Show that if $\mathbf{x}(t)$ is a solution of system (I), then $\mathbf{c}(t) = e^{kt}\mathbf{x}(t)$ is a solution of system (II).

In the following exercises, (a) solve the system with the given initial value and (b) sketch rough phase portraits for the dynamical systems (or use the Java tool to sketch the underlying vector fields and some trajectories).

26, 32. $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

28, 34. $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 4 & 3 \\ 4 & 8 \end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

29, 35. $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

31. $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

[Sketching the phase portrait is optional for this one.]

42. Consider the interaction of two species of animals in a habitat. We are told that the change of the populations $\mathbf{x}(t)$ and $\mathbf{y}(t)$ can be modeled by the equations

$$\left\{ \begin{matrix} \dfrac{dx}{dt} = 1.4x - 1.2y \\ \dfrac{dy}{dt} = 0.8x - 1.4y \end{matrix} \right\}$$

where time t is measured in years.

a. What kind of interaction do we observe (symbiosis, competition, or predator-prey)?

b. Sketch a phase portrait for this system. From the nature of the problem, we are interested only in the first quadrant.

c. What will happen in the long term? Does the outcome depend on the initial populations? If so, how?

52. Find all solutions of the system $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\mathbf{x}$, where λ is an arbitrary constant. *Hint*: Exercise 21 and 24 are helpful. Sketch a phase portrait. For which choices of λ is the zero state a stable equilibrium solution?

Section 9.2:

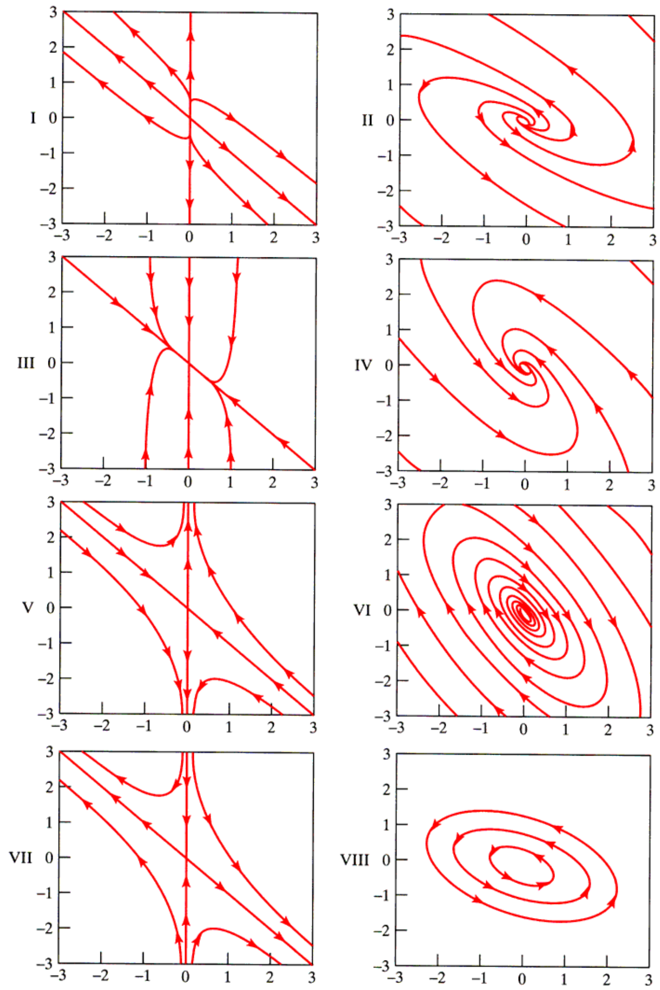
6. Find all complex solutions of the system $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}\mathbf{x}$ in the form given in Theorem 9.2.3.

What solution do you get if you let $c_1 = c_2 = 1$? [**<u>Note</u>**: The coefficients here are for the expression $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$ with complex eigenvalues λ_1 and λ_2 and corresponding complex eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. The problem is ambiguously worded in that there are infinitely many choices for $\mathbf{v}_1$ and $\mathbf{v}_2$ and for any given initial conditions, these would yield different values for c_1 and c_2.]

7. Determine the stability of the system $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}\mathbf{x}$.

12. Determine the stability of the system $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -2 \end{bmatrix}\mathbf{x}$.

For each of the linear systems in Exercises 22 through 26, find the matching phase portrait (to the right).



22. $\mathbf{x}(t+1)=\begin{bmatrix}3 & 0\\ -2.5 & 0.5\end{bmatrix}\mathbf{x}(t)$

23. $\mathbf{x}(t+1)=\begin{bmatrix}-1.5 & -1\\ 2 & 0.5\end{bmatrix}\mathbf{x}(t)$

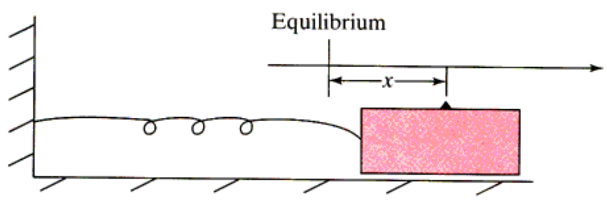
24. $\dfrac{d\mathbf{x}}{dt}=\begin{bmatrix}3 & 0\\ -2.5 & 0.5\end{bmatrix}\mathbf{x}$

25. $\dfrac{d\mathbf{x}}{dt}=\begin{bmatrix}-1.5 & -1\\ 2 & 0.5\end{bmatrix}\mathbf{x}$

26. $\dfrac{d\mathbf{x}}{dt}=\begin{bmatrix}-2 & 0\\ 3 & 1\end{bmatrix}\mathbf{x}$

31. Solve the system $\dfrac{d\mathbf{x}}{dt}=\begin{bmatrix}-1 & -2\\ 2 & -1\end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0)=\begin{bmatrix}1\\ -1\end{bmatrix}$. Give the solution in real form. Sketch the solution.

36. Consider the following mass-spring system:



Let $x(t)$ be the deviation of the block from the equilibrium position at time t. Consider the velocity $v(t)=\dfrac{dx}{dt}$ of the block. There are two forces acting on the mass: The spring force F_s, which is assumed to be proportional to the displacement x, and the force F_f of friction, which is assumed to be proportional to the velocity

$$F_s=-px\ , \quad F_f==-qv$$

where $p>0$ and $q\geq 0$. (q is 0 if the oscillation is frictionless.) Therefore the total force acting on the mass is

$$F=F_s+F_f=-px-qv\ .$$

By Newton's second law of motion, we have

$$F = ma = m\frac{dv}{dt},$$

where a represents acceleration and m the mass of the block. Combining the last two equations, we find that

$$m\frac{dv}{dt} = -px - qv \quad \text{or} \quad \frac{dv}{dt} = -\frac{p}{m}x - \frac{q}{m}v.$$

Let $b = \frac{p}{m}$ and $c = \frac{q}{m}$ for simplicity. Then the dynamics of this mass-spring system are described by the system

$$\begin{Bmatrix} \frac{dx}{dt} = & v \\ \frac{dv}{dt} = & -bx - cv \end{Bmatrix} \quad (b > 0,\ c \geq 0).$$

Sketch a phase portrait for this system in each of the following cases, and describe briefly the significance of your trajectories in terms of the movement of the block. Comment on the stability in each case.

a. $c = 0$ (frictionless). Find the period. b. $c^2 < 4b$ (underdamped). c. $c^2 > 4b$ (overdamped).

39. Solve the system

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}\mathbf{x}.$$

Compare this with Exercise 9.1.24. When is the zero state a stable equilibrium solution?

For additional practice:

Section 9.1:

Solve the following initial value problems and graph the solution:

4. $\frac{dy}{dt} = 0.8t$ with $y(0) = -0.8$ 5. $\frac{dy}{dt} = 0.8y$ with $y(0) = -0.8$

13. In 1778, a wealthy Pennsylvanian merchant named Jacob DeHaven lent \$450,000 to the Continental Congress to support the troops at Valley Forge. The loan was never repaid. Mr. DeHaven's descendants are taking the U.S. government to court to collect what they believe they are owed. The going interest rate at the time was 6%. How much were the DeHaven's owed in 1990
a. if interest is compounded yearly?
b. if interest is compounded continuously?
(Adapted from *The New York Times*, May 27, 1990.)

21. Consider the system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ with $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Sketch a direction field for $\mathbf{A}\mathbf{x}$ (or use the Java tool). Based on your sketch, describe the trajectories geometrically. Can you find the solutions analytically?

22. Consider a linear system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ of arbitrary size. Suppose $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions of this system. Is the sum $\mathbf{x}(t) = \mathbf{x}_1(t) + \mathbf{x}_2(t)$ a solution as well? How do you know?

23. . Consider a linear system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ of arbitrary size. Suppose $\mathbf{x}_1(t)$ is a solution of this system and k is an arbitrary constant. Is $\mathbf{x}(t) = k\,\mathbf{x}_1(t)$ a solution as well? How do you know?

43. Two herds of vicious animals are fighting each other to the death. During the fight, the populations $x(t)$ and $y(t)$ of the two species can be modeled by the following system:

$$\begin{Bmatrix} \dfrac{dx}{dt} = & -4y \\ \dfrac{dy}{dt} = -x & \end{Bmatrix}$$

a. What is the significance of the constants –4 and –1 in these equations? Which species has the more vicious (or more efficient) fighters?

b. Sketch a phase portrait for this system.

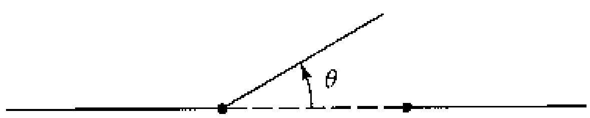
c. Who wins the fight (in the sense that some individuals of that species are left while the other herd is eradicated)? How does your answer depend on the initial populations?

49. Here is a continuous model of a person's glucose regulatory system. (Compare this with Exercise 7.1.52.) Let $g(t)$ and $h(t)$ be the excess glucose and insulin concentrations in a person's blood. We are told that

$$\begin{Bmatrix} \dfrac{dg}{dt} = -g - 0.2h \\ \dfrac{dh}{dt} = 0.6g - 0.2h \end{Bmatrix}$$

where time t is measured in hours. After a heavy holiday dinner, we measure $g(0) = 30$ and $h(0) = 0$. Find closed formulas for $g(t)$ and $h(t)$. Sketch the trajectory.

54. Consider a door that opens to only one side (as most doors do). A spring mechanism closes the door automatically. The state of the door at any given time t (measured in seconds) is determined by the angular displacement $\theta(t)$ (measured in radians) and the angular velocity $\omega(t) = \dfrac{d\theta}{dt}$. Note that θ is always positive or zero (since the door opens to only one side), but ω can be positive or negative (depending on whether the door is opening or closing).



When the door is moving freely (nobody is pushing or pulling), its movement is subject to the following differential equations:

$$\begin{Bmatrix} \dfrac{d\theta}{dt} = & \omega & \text{(the definition of } \omega) \\ \dfrac{d\omega}{dt} = -2\theta - 3\omega & & \text{(-}2\theta \text{ reflects the force of the spring, and -}3\omega \text{ models friction)} \end{Bmatrix}$$

a. Sketch the phase portrait of this system.

b. Discuss the movement of the door represented by the qualitatively different trajectories. For which initial states does the door slam (i,e,, reach $\theta = 0$ with velocity $\omega < 0$)?

55. Answer the questions posed in Exercise 54 for the system $\begin{Bmatrix} \dfrac{d\theta}{dt} = & \omega \\ \dfrac{d\omega}{dt} = -p\theta - q\omega \end{Bmatrix}$

where p and q are positive, and $q^2 > 4p$.

Section 9.2:

34. Solve the system $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 7 & 10 \\ -4 & -5 \end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Give the solution in real form. Sketch the solution.