Math E-21b – Spring 2024 – Homework #12

Problem 1. Consider a linear transformation L from $\mathbf{R}^m$ to $\mathbf{R}^n$. Show that there is an orthonormal basis $\{\mathbf{v}_1,\dots,\mathbf{v}_m\}$ of $\mathbf{R}^m$ such that the vectors $\{L(\mathbf{v}_1),\dots,L(\mathbf{v}_m)\}$ are orthogonal. Note that some of the vectors $L(\mathbf{v}_i)$ may be zero. *Hint*: Consider an orthonormal basis $\{\mathbf{v}_1,\dots,\mathbf{v}_m\}$ for the symmetric matrix $\mathbf{A}^\mathrm{T}\mathbf{A}$.

Problem 2. Consider a linear transformation T from $\mathbf{R}^m$ to $\mathbf{R}^n$, where $m \le n$. Show that there is an orthonormal basis $\{\mathbf{v}_1,\dots,\mathbf{v}_m\}$ of $\mathbf{R}^m$ and an orthonormal basis $\{\mathbf{w}_1,\dots,\mathbf{w}_n\}$ of $\mathbf{R}^n$ such that $T(\mathbf{v}_i)$ is a scalar multiple of $\mathbf{w}_i$, for $i = 1,\dots,m$. *Hint*: Problem 1 is helpful.

Problem 3. Consider a symmetric matrix **A**. If the vector **v** is in the image of **A** and **w** is in the kernel of **A**, is **v** necessarily orthogonal to **w**? Justify your answer.

Problem 4. Consider a symmetric $n \times n$ matrix **A** with $\mathbf{A}^2 = \mathbf{A}$. Is the linear transformation $T(\mathbf{x}) = \mathbf{Ax}$ necessarily the orthogonal projection onto a subspace of $\mathbf{R}^n$?

Problem 5. Determine the definiteness of the quadratic form $q(x,y) = 6x^2 + 4xy + 3y^2$.

Problem 6. Determine the definiteness of the quadratic form $q(x,y) = 2x^2 + 6xy + 4y^2$.

Problem 7. If **A** is a symmetric matrix, what can you say about the definiteness of $\mathbf{A}^2$? When is $\mathbf{A}^2$ positive definite?

Problem 8. Recall that a real square matrix **A** is called *skew-symmetric* if $\mathbf{A}^\mathrm{T} = -\mathbf{A}$.

a. If **A** is skew-symmetric, is $\mathbf{A}^2$ skew-symmetric as well? Or is $\mathbf{A}^2$ symmetric?

b. If **A** is skew-symmetric, what can you say about the definiteness of $\mathbf{A}^2$? What about the eigenvalues of $\mathbf{A}^2$?

c. What can you say about the complex eigenvalues of a skew-symmetric matrix? Which skew-symmetric matrices are diagonalizable over **R** (the real numbers)?

Problem 9. If **A** is an invertible symmetric matrix, what is the relationship between the definiteness of **A** and $\mathbf{A}^{-1}$?

Sketch the curves in Problems 10-12. In each case, draw and label the principal axes, label the intercepts of the curve with the principal axes, and give the formula of the curve in the coordinate system (u,v) defined by the principal axes.

Problem 10. $q(x,y) = xy = 1$

Problem 11. (8.2/18) $q(x,y) = 9x^2 - 4xy + 6y^2 = 1$

Problem 12. $q(x,y) = x^2 + 4xy + 4y^2 = 1$

Problem 13. On the surface $-x^2 + y^2 - z^2 + 10xz = 1$, find the two points closest to the origin.

Problem 14. (<u>**SVD Problem**</u> – Section 8.3) a) Given the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, find an orthonormal basis $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ for the domain and an orthonormal basis $\{\mathbf{w}_1,\mathbf{w}_2\}$ for the codomain such that the images of the basis vectors of the domain $\{\mathbf{Au}_1,\mathbf{Au}_2,\mathbf{Au}_3\}$ are scalar multiples of the orthonormal basis vectors $\{\mathbf{w}_1,\mathbf{w}_2\}$ (or the zero vector **0**). [<u>Reference</u>: Problems 1-2 above].

b) Use these to find the singular value decomposition $\mathbf{A} = \mathbf{Q\Sigma P}^\mathrm{T}$ where **Q** is a 2×2 orthogonal matrix, **P** is a 3×3 orthogonal matrix, and $\mathbf{\Sigma}$ is a 2×3 matrix that contains the "singular values" associated with the given matrix (as described in the text or Lecture Notes #12).

c) Repeat the above exercise for the matrix $\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$. This is the transpose of the previous matrix.

[<u>Note</u>: There is a very simple way to answer (c) without additional calculation!]

For additional practice:
Section 8.2:

1. For the quadratic form $q(x_1,x_2)=6x_1^2-7x_1x_2+8x_2^2$, find a symmetric matrix **A** such that $q(\mathbf{x})=\mathbf{x}\cdot\mathbf{A}\mathbf{x}=\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$.
2. For the quadratic form $q(x_1,x_2)=x_1x_2$, find a symmetric matrix **A** such that $q(\mathbf{x})=\mathbf{x}\cdot\mathbf{A}\mathbf{x}=\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$.
3. For the quadratic form $q(x_1,x_2,x_3)=3x_1^2+4x_2^2+5x_3^2+6x_1x_3+7x_2x_3$, find a symmetric matrix **A** such that $q(\mathbf{x})=\mathbf{x}\cdot\mathbf{A}\mathbf{x}=\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$.

Sketch the curves in Exercises 15. Draw and label the principal axes, label the intercepts of the curve with the principal axes, and give the formula of the curve in the coordinate system defined by the principal axes.

15. $q(x_1,x_2)=6x_1^2+4x_1x_2+3x_2^2=1$

21. a. Sketch the following three surfaces:

$$\begin{aligned} x_1^2+4x_2^2+9x_3^2&=1\\ x_1^2+4x_2^2-9x_3^2&=1\,.\\ -x_1^2-4x_2^2+9x_3^2&=1 \end{aligned}$$

Which of these are bounded? Which are connected? Label the points closest to and farthest from the origin (if there are any).

b. Consider the surface $q(x_1,x_2,x_3)=x_1^2+2x_2^2+3x_3^2+x_1x_2+2x_1x_3+3x_2x_3=1$.

Which of the three surfaces in part (a) does this surface qualitatively resemble most? Which points on this surface are closest to the origin? Give a rough approximation. You may use technology.

Chapter 8 True/False questions (work with real numbers throughout)

1. If A is an orthogonal matrix, then there must exist a symmetric invertible matrix S such that $S^{-1}AS$ is diagonal.
2. The singular value of the 2×1 matrix $\begin{bmatrix}3\\4\end{bmatrix}$ is 5.
3. The function $q(x_1,x_2)=3x_1^2+4x_1x_2+5x_2$ is a quadratic form.
4. The singular values of any matrix A are the eigenvalues of matrix A^TA.
5. If matrix A is positive definite, then all the eigenvalues of A must be positive.
6. The function $q(\vec{x})=\vec{x}^T\begin{bmatrix}1&2\\2&4\end{bmatrix}\vec{x}$ is a quadratic form.
7. The singular values of any diagonal matrix D are the absolute values of the diagonal entries of D.
8. The equation $2x^2+5xy+3y^2=1$ defines an ellipse.
9. All symmetric matrices are diagonalizable.
10. If the matrix $\begin{bmatrix}a&b\\b&c\end{bmatrix}$ is positive definite, then a must be positive.
11. If the singular values of a 2×2 matrix A are 3 and 4, then there must exist a unit vector $\vec{u}$ in $\mathbb{R}^2$ such that $\|A\vec{u}\|=4$.
12. The determinant of a negative definite 4×4 matrix must be positive.
13. If A is a symmetric matrix such that $A\vec{v}=3\vec{v}$ and $A\vec{w}=4\vec{w}$, then the equation $\vec{v}\cdot\vec{w}=0$ must hold.
14. Matrix $\begin{bmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{bmatrix}$ is negative definite.
15. All skew-symmetric matrices are diagonalizable (over $\mathbb{R}$).
16. If A is any matrix, then matrix AA^T is diagonalizable.
17. All positive definite matrices are invertible.
18. Matrix $\begin{bmatrix}3&2&1\\2&3&2\\1&2&3\end{bmatrix}$ is diagonalizable.
19. The singular values of any triangular matrix are the absolute values of its diagonal entries.
20. If A is any matrix, then matrix A^TA is the transpose of AA^T.

21. If $\vec{v}$ and $\vec{w}$ are linearly independent eigenvectors of a symmetric matrix A, then $\vec{w}$ must be orthogonal to $\vec{v}$.

22. For any $n \times m$ matrix A there exists an orthogonal $m \times m$ matrix S such that the columns of matrix AS are orthogonal.

23. If A is a symmetric $n \times n$ matrix such that $A^n = 0$, then A must be the zero matrix.

24. If $q(\vec{x})$ is a positive definite quadratic form, then so is $kq(\vec{x})$, for any scalar k.

25. If A is an invertible symmetric matrix, then A^2 must be positive definite.

26. If the two columns $\vec{v}$ and $\vec{w}$ of a 2×2 matrix A are orthogonal, then the singular values of A must be $\|\vec{v}\|$ and $\|\vec{w}\|$.

27. If A and S are invertible $n \times n$ matrices, then matrices A and $S^T AS$ must be similar.

28. If A is negative definite, then all the diagonal entries of A must be negative.

29. If the positive definite matrix A is similar to the symmetric matrix B, then B must be positive definite as well.

30. If A is a symmetric matrix, then there must exist an orthogonal matrix S such that SAS^T is diagonal.

31. If A and B are 2×2 matrices, then the singular values of matrices AB and BA must be the same.

32. If A is any orthogonal matrix, then matrix $A + A^{-1}$ is diagonalizable (over $\mathbb{R}$).

33. The product of two quadratic forms in 3 variables must be a quadratic form as well.

34. The function $q(\vec{x}) = \vec{x}^T \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x}$ is a quadratic form.

35. If the determinants of all the principal submatrices of a symmetric 3×3 matrix A are negative, then A must be negative definite.

36. If A and B are positive definite $n \times n$ matrices, then matrix $A + B$ must be positive definite as well.

37. If A is a positive definite $n \times n$ matrix and $\vec{x}$ is a nonzero vector in $\mathbb{R}^n$, then the angle between $\vec{x}$ and $A\vec{x}$ must be acute.

38. If the 2×2 matrix A has the singular values 2 and 3 and the 2×2 matrix B has the singular values 4 and 5, then both singular values of matrix AB must be ≤ 15.

39. The equation $A^T A = AA^T$ holds for all square matrices A.

40. For every symmetric $n \times n$ matrix A there exists a constant k such that $A + kI_n$ is positive definite.

41. If matrix $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ is positive definite, then af must exceed c^2.

42. If A is positive definite, then all the entries of A must be positive or zero.

43. If A is indefinite, then 0 must be an eigenvalue of A.

44. If A is a 2×2 matrix with singular values 3 and 5, then there must exist a unit vector $\vec{u}$ in $\mathbb{R}^2$ such that $\|A\vec{u}\| = 4$.

45. If A is skew symmetric, then A^2 must be negative semidefinite.

46. The product of the n singular values of an $n \times n$ matrix A must be $|\det A|$.

47. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, then there exist exactly 4 orthogonal 2×2 matrices S such that $S^{-1}AS$ is diagonal.

48. The sum of two quadratic forms in 3 variables must be a quadratic form as well.

49. The eigenvalues of a symmetric matrix A must be equal to the singular values of A.

50. Similar matrices must have the same singular values.

51. If A is a symmetric 2×2 matrix with eigenvalues 1 and 2, then the angle between $\vec{x}$ and $A\vec{x}$ must be less than $\pi/6$, for all nonzero vectors $\vec{x}$ in $\mathbb{R}^2$.

52. If both singular values of a 2×2 matrix A are less than 5, then all the entries of A must be less than 5.

53. If A is a positive definite matrix, then the largest entry of A must be on the diagonal.

54. If A and B are real symmetric matrices such that $A^3 = B^3$, then A must be equal to B.