Math E-21b – Spring 2025 – Homework #11

Problem 1. Determine whether the zero state is a stable equilibrium of the dynamical system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$,

where $\mathbf{A} = \begin{bmatrix} 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 \end{bmatrix}$, i.e. do all trajectories approach the equilibrium (zero) state?

[Note: Zero state refers to the case where $\mathbf{x}(0) = \mathbf{0}$.]

Problem 2. Given the matrix $\mathbf{A} = \begin{bmatrix} 0.6 & k \\ -k & 0.6 \end{bmatrix}$, for which real numbers k is the zero state a stable equilibrium of the dynamical system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$?

Problem 3. For the matrix $\mathbf{A} = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}$, find real closed formulas for the trajectory $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$,

where $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Draw a rough sketch.

Problem 4. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & -3 \\ 1.2 & -2.6 \end{bmatrix}$, find real closed formulas for the trajectory $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$,

where $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Draw a rough sketch.

Problem 5. It's easy to see that if $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $\mathbf{A}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{A}^t = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for all integers $t > 1$.

Similarly, if $\mathbf{A} = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$, then $\mathbf{A}^2 = \begin{bmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $\mathbf{A}^t = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ for all integers $t > 2$.

More generally, if $\mathbf{A}$ is any $n \times n$ is any upper triangular matrix with all 0's on the main diagonal, we have that $\mathbf{A}^t = [\mathbf{0}]$ (the $n \times n$ zero matrix) for $t \geq n-1$. Such a matrix is called ***nilpotent***.

(a) We have shown that if a linear transformation given by a 2×2 matrix has an eigenvalue λ with algebraic multiplicity 2 and geometric multiplicity 1, then a basis can be found such that relative to this basis the matrix of this transformation can be put in the normal form $\mathbf{B} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

By writing $\mathbf{B} = \lambda\mathbf{I} + \mathbf{A}$ where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is nilpotent, calculate $\mathbf{B}^t$ for all positive integers t.

(b) Given the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$, find $\mathbf{x}(t) = \mathbf{A}^t\mathbf{x}_0$ for all t if $\mathbf{x}_0 = \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Problem 6. (a) Similarly, if a linear transformation given by a 3×3 matrix has an eigenvalue λ with algebraic multiplicity 3 and geometric multiplicity 1, then a basis can be found such that relative to this basis the matrix of this transformation can be put in the normal form $\mathbf{B} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$. By writing $\mathbf{B} = \lambda\mathbf{I} + \mathbf{A}$ where

$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is nilpotent, calculate $\mathbf{B}^t$ for all positive integers t. [***Hint***: Binomial Theorem applies.]

(b) For the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 3 & 2 \\ -2 & 1 & 3 \end{bmatrix}$, find $\mathbf{x}(t) = \mathbf{A}^t\mathbf{x}_0$ for all t if $\mathbf{x}_0 = \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Problem 7. Consider an affine transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{b}$ is a vector in $\mathbf{R}^n$. (Compare this with Exercise 7.3.45.) Suppose that 1 is not an eigenvalue of $\mathbf{A}$.

a. Find the vector $\mathbf{v}$ in $\mathbf{R}^n$ such that $T(\mathbf{v}) = \mathbf{v}$; this vector is called the equilibrium state of the dynamical system $\mathbf{x}(t+1) = T(\mathbf{x}(t))$.

b. When is the equilibrium $\mathbf{v}$ in part (a) stable (meaning that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{v}$ for all trajectories)?

For problems 8-10, <u>without using technology</u>, find an orthonormal eigenbasis for the given matrix:

Problem 8. $\mathbf{A} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$ **Problem 9.** $\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **Problem 10.** $\mathbf{A} = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$.

Problem 11. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}$, find an orthogonal matrix $\mathbf{S}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$. Do not use technology.

Problem 12. Let L from $\mathbf{R}^3$ to $\mathbf{R}^3$ be the reflection about the line spanned by $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

a. Find an orthonormal eigenbasis $\mathscr{B}$ for L.
b. Find the matrix $\mathbf{B}$ of L with respect to $\mathscr{B}$.
c. Find the matrix $\mathbf{A}$ of L with respect to the standard basis of $\mathbf{R}^3$.

Problem 13. If $\mathbf{A}$ is invertible and orthogonally diagonalizable, is $\mathbf{A}^{-1}$ orthogonally diagonalizable as well?

Problem 14. a. Find the eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ with their multiplicities.

Note that the algebraic multiplicity agrees with the geometric multiplicity. (Why?) *Hint*: What is $\ker(\mathbf{A})$?

b. Find the eigenvalues of the matrix $\mathbf{B} = \begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix}$ with their multiplicities. Do not use technology.

c. Use your result in part (b) to find det($\mathbf{B}$).

Problem 15. Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Find an orthonormal eigenbasis for $\mathbf{A}$.

For additional practice:
Section 7.6:
For the matrices in Exercises 1-4, determine whether the zero state is a stable equilibrium of the dynamical system $\mathbf{x}(t+1)=\mathbf{A}\mathbf{x}(t)$.

1. $\mathbf{A}=\begin{bmatrix}0.9 & 0\\ 0 & 0.8\end{bmatrix}$ 2. $\mathbf{A}=\begin{bmatrix}-1.1 & 0\\ 0 & 0.9\end{bmatrix}$ 3. $\mathbf{A}=\begin{bmatrix}0.8 & 0.7\\ -0.7 & 0.8\end{bmatrix}$ 4. $\mathbf{A}=\begin{bmatrix}-0.9 & -0.4\\ 0.4 & -0.9\end{bmatrix}$

37. Consider the national income of a country, which consists of consumption, investment, and government expenditures. Here we assume the government expenditure to be constant, at G_0, while the national income $Y(t)$, consumption $C(t)$, and investment $I(t)$ change over time. According to a simple model, we have:

$$\left\{\begin{array}{l} Y(t)=C(t)+I(t)+G_0\\ C(t+1)=\gamma Y(t)\\ I(t+1)=\alpha[C(t+1)-C(t)]\end{array}\right\}\quad \begin{array}{l}(0<\gamma<1),\\ (\alpha>0)\end{array}$$

where γ is the marginal propensity to consume and α is the acceleration coefficient. (See Paul Samuelson, “Interactions between the Multiplier Analysis and the Principle of Acceleration,” *Review of Economic Statistics*, May 1939, pp. 75-78.)

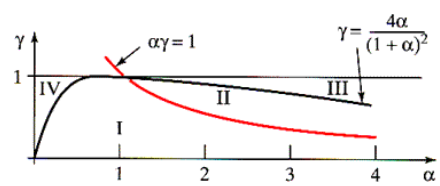
a. Find the equilibrium solution of these equations, i.e., when $Y(t+1)=Y(t)$, $C(t+1)=C(t)$, and $I(t+1)=I(t)$.

b. Let $y(t)$, $c(t)$, and $i(t)$ be the deviations of $Y(t)$, $C(t)$, and $I(t)$, respectively, from the equilibrium state you found in part (a). These quantities are related by the equations $\left\{\begin{array}{l} y(t)=c(t)+i(t)\\ c(t+1)=\gamma y(t)\\ i(t+1)=\alpha[c(t+1)-c(t)]\end{array}\right\}$. (Verify this!) By substituting $y(t)$ into the second equation, set up equations of the form $\left\{\begin{array}{l} c(t+1)=p\,c(t)+q\,i(t)\\ i(t+1)=r\,c(t)+s\,i(t)\end{array}\right\}$.

c. When $\alpha=5$ and $\gamma=0.2$, determine the stability of the zero state of this system.

d. When $\alpha=1$ (and γ is arbitrary, $0<\gamma<1$), determine the stability of the zero state.

e. For each of the four sectors in the $\alpha\gamma$-plane, determine the stability of the zero state. Discuss the various cases, in practical terms.



40. Consider the matrix $\mathbf{A}=\begin{bmatrix}p & -q & -r & -s\\ q & p & s & -r\\ r & -s & p & q\\ s & r & -q & p\end{bmatrix}$ where *p*, *q*, *r*, *s* are arbitrary real numbers.

(Compare this with Exercise 5.3.64.)

a. Compute $\mathbf{A}^T\mathbf{A}$.

b. For which values of *p*, *q*, *r*, *s* is **A** invertible? Find the inverse if it exists.

c. Find the determinant of **A**.

d. Find the complex eigenvalues of **A**.

e. If **x** is a vector in $\mathbf{R}^4$, what is the relationship between $\|\mathbf{x}\|$ and $\|\mathbf{A}\mathbf{x}\|$?

f. Consider the numbers $59=3^2+3^2+4^2+5^2$ and $37=1^2+2^2+4^2+4^2$. Express the number 2183 as the sum of squares of four integers: $2183=a^2+b^2+c^2+d^2$.
Hint: Part (e) is useful. Note that $2183=59\cdot 37$.

g. The French mathematician Joseph-Louis Lagrange (1736–1813) showed that any prime number can be expressed as the sum of squares of four integers. Using this fact and your work in part (f) as a guide, show that any positive integer can be expressed in this way.