

Math E-21b – Spring 2019 – Homework #11

Section 7.6:

10. Determine whether the zero state is a stable equilibrium of the dynamical system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$,

where $\mathbf{A} = \begin{bmatrix} 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 \end{bmatrix}$. [Note: Zero state refers to the case where $\mathbf{x}(0) = \mathbf{0}$.]

12. Given the matrix $\mathbf{A} = \begin{bmatrix} 0.6 & k \\ -k & 0.6 \end{bmatrix}$, for which real numbers k is the zero state a stable equilibrium of the dynamical system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$?

17. For the matrix $\mathbf{A} = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}$, find real closed formulas for the trajectory $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$,

where $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Draw a rough sketch.

18. For the matrix $\mathbf{A} = \begin{bmatrix} -0.8 & 0.6 \\ -0.8 & -0.8 \end{bmatrix}$, find real closed formulas for the trajectory $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$,

where $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Draw a rough sketch. [Note: The solution may be awkward, but it's doable.]

24. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & -3 \\ 1.2 & -2.6 \end{bmatrix}$, find real closed formulas for the trajectory $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$,

where $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Draw a rough sketch.

38. Consider an affine transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, where \mathbf{A} is an $n \times n$ matrix and \mathbf{b} is a vector in \mathbf{R}^n . (Compare this with Exercise 7.3.45.) Suppose that 1 is not an eigenvalue of \mathbf{A} .

a. Find the vector \mathbf{v} in \mathbf{R}^n such that $T(\mathbf{v}) = \mathbf{v}$; this vector is called the equilibrium state of the dynamical system $\mathbf{x}(t+1) = T(\mathbf{x}(t))$.

b. When is the equilibrium \mathbf{v} in part (a) stable (meaning that $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{v}$ for all trajectories)?

Section 8.1:

6. Without using technology, find an orthonormal eigenbasis for the matrix $\mathbf{A} = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$.

10. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}$, find an orthogonal matrix \mathbf{S} and a diagonal matrix \mathbf{D}

such that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$. Do not use technology.

12. Let L from \mathbf{R}^3 to \mathbf{R}^3 be the reflection about the line spanned by $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

a. Find an orthonormal eigenbasis \mathcal{B} for L .

b. Find the matrix \mathbf{B} of L with respect to \mathcal{B} .

c. Find the matrix \mathbf{A} of L with respect to the standard basis of \mathbf{R}^3 .

16. a. Find the eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ with their multiplicities. Note that the algebraic

multiplicity agrees with the geometric multiplicity. (Why?) *Hint:* What is the kernel of \mathbf{A} ?

b. Find the eigenvalues of the matrix $\mathbf{B} = \begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix}$ with their multiplicities. Do not use technology.

c. Use your result in part (b) to find $\det(\mathbf{B})$.

24. Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Find an orthonormal eigenbasis for \mathbf{A} .

For additional practice:

Section 7.6:

For the matrices in Exercises 1-4, determine whether the zero state is a stable equilibrium of the dynamical system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$.

1. $\mathbf{A} = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.8 \end{bmatrix}$ 2. $\mathbf{A} = \begin{bmatrix} -1.1 & 0 \\ 0 & 0.9 \end{bmatrix}$ 3. $\mathbf{A} = \begin{bmatrix} 0.8 & 0.7 \\ -0.7 & 0.8 \end{bmatrix}$ 4. $\mathbf{A} = \begin{bmatrix} -0.9 & -0.4 \\ 0.4 & -0.9 \end{bmatrix}$

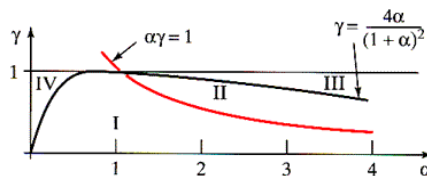
37. Consider the national income of a country, which consists of consumption, investment, and government expenditures. Here we assume the government expenditure to be constant, at G_0 , while the national income $Y(t)$, consumption $C(t)$, and investment $I(t)$ change over time. According to a simple model, we have:

$$\left\{ \begin{array}{l} Y(t) = C(t) + I(t) + G_0 \\ C(t+1) = \gamma Y(t) \\ I(t+1) = \alpha [C(t+1) - C(t)] \end{array} \right\} \quad \begin{array}{l} (0 < \gamma < 1), \\ (\alpha > 0) \end{array} \quad \begin{array}{l} \text{where } \gamma \text{ is the marginal propensity to consume and } \alpha \text{ is the} \\ \text{acceleration coefficient. (See Paul Samuelson, "Interactions} \\ \text{between the Multiplier Analysis and the Principle of Acceleration,"} \\ \text{Review of Economic Statistics, May 1939, pp. 75-78.)} \end{array}$$

- a. Find the equilibrium solution of these equations, i.e., when $Y(t+1) = Y(t)$, $C(t+1) = C(t)$, and $I(t+1) = I(t)$.
- b. Let $y(t)$, $c(t)$, and $i(t)$ be the deviations of $Y(t)$, $C(t)$, and $I(t)$, respectively, from the equilibrium state you found in part (a). These quantities are related by the equations $\left\{ \begin{array}{l} y(t) = c(t) + i(t) \\ c(t+1) = \gamma y(t) \\ i(t+1) = \alpha [c(t+1) - c(t)] \end{array} \right\}$. (Verify this!)

By substituting $y(t)$ into the second equation, set up equations of the form $\left\{ \begin{array}{l} c(t+1) = p c(t) + q i(t) \\ i(t+1) = r c(t) + s i(t) \end{array} \right\}$.

- c. When $\alpha = 5$ and $\gamma = 0.2$, determine the stability of the zero state of this system.
- d. When $\alpha = 1$ (and γ is arbitrary, $0 < \gamma < 1$), determine the stability of the zero state.
- e. For each of the four sectors in the $\alpha\gamma$ -plane, determine the stability of the zero state. Discuss the various cases, in practical terms.



40. Consider the matrix $\mathbf{A} = \begin{bmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{bmatrix}$ where p, q, r, s are arbitrary real numbers. (Compare this with

Exercise 5.3.64.)

a. Compute $\mathbf{A}^T \mathbf{A}$.

b. For which values of p, q, r, s is \mathbf{A} invertible? Find the inverse if it exists.

c. Find the determinant of \mathbf{A} .

d. Find the complex eigenvalues of \mathbf{A} .

e. If \mathbf{x} is a vector in \mathbf{R}^4 , what is the relationship between $\|\mathbf{x}\|$ and $\|\mathbf{A}\mathbf{x}\|$?

f. Consider the numbers $59 = 3^2 + 3^2 + 4^2 + 5^2$ and $37 = 1^2 + 2^2 + 4^2 + 4^2$. Express the number 2183 as the sum of squares of four integers: $2183 = a^2 + b^2 + c^2 + d^2$.

Hint: Part (e) is useful. Note that $2183 = 59 \cdot 37$.

g. The French mathematician Joseph-Louis Lagrange (1736–1813) showed that any prime number can be expressed as the sum of squares of four integers. Using this fact and your work in part (f) as a guide, show that any positive integer can be expressed in this way.

Section 8.1:

3. Without using technology, find an orthonormal eigenbasis for the matrix $\mathbf{A} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$.

5. Without using technology, find an orthonormal eigenbasis for the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

15. If \mathbf{A} is invertible and orthogonally diagonalizable, is \mathbf{A}^{-1} orthogonally diagonalizable as well?