Math E-21b – Spring 2024 – Homework #10

Problem 1. If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, prove that $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$.
[*Hint*: Try writing it out in index notation.]

Problem 2. If matrix $\mathbf{A}$ is similar to matrix $\mathbf{B}$, prove that $\operatorname{tr}(\mathbf{A}) = \operatorname{tr}(\mathbf{B})$. [*Hint*: Problem 1 is helpful.]

Problem 3. Consider an eigenvalue λ of an $n \times n$ matrix $\mathbf{A}$. We know that λ is an eigenvalue of $\mathbf{A}^{\mathrm{T}}$ as well (since $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$ have the same characteristic polynomial). Compare the geometric multiplicities of λ as an eigenvalue of $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$.

Problem 4. Suppose that $\mathbf{B} = \mathbf{S}^{-1}\mathbf{AS}$ for some $n \times n$ matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{S}$, i.e. that $\mathbf{A}$ and $\mathbf{B}$ are *similar*.

a. Explain why if $\mathbf{x}$ is in $\ker(\mathbf{B})$, then $\mathbf{Sx}$ is in $\ker(\mathbf{A})$.

b. Explain why the linear transformation $T(\mathbf{x}) = \mathbf{Sx}$ from $\ker(\mathbf{B})$ to $\ker(\mathbf{A})$ is an isomorphism.

c. Explain why $\operatorname{nullity}(\mathbf{A}) = \operatorname{nullity}(\mathbf{B})$ and $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{B})$.

Problem 5. Is matrix $\begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$ similar to $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$? [You **must** provide an explanation.]

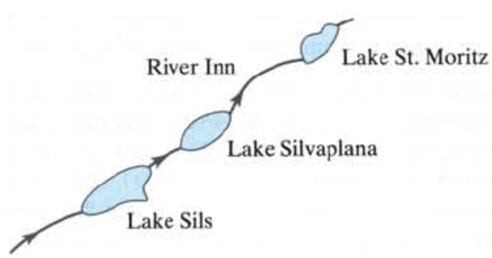
Problem 6. Is the matrix $\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$ similar to the matrix $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$? [You **must** provide an explanation.]

Problem 7. If $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrices with the same trace and determinant, are they necessarily similar?

Problem 8. In an unfortunate accident involving an Austrian truck, 100 kg of a highly toxic substance are spilled into Lake Sils, in the Swiss Engadine Valley. The river Inn carries the pollutant down to Lake Silvaplana and later to Lake St. Moritz.

This sorry state, *t* weeks after the accident, can be described by the vector

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \left.\begin{matrix} \text{pollutant in Lake Sils} \\ \text{pollutant in Lake Silvaplana} \\ \text{pollutant in Lake St. Moritz} \end{matrix}\right\} \text{(in kg)}.$$



Suppose that $\mathbf{x}(t+1) = \begin{bmatrix} 0.7 & 0 & 0 \\ 0.1 & 0.6 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix} \mathbf{x}(t)$.

a. Explain the significance of the entries of the transformation matrix in practical terms.

b. Find <u>closed formulas</u> for the amount of pollutant in each of the three lakes *t* weeks after the accident. Graph the three functions against time (on the same axes). When does the pollution in Lake Silvaplana reach a maximum?

In Problems 9-11, determine if $\mathbf{A}$ is diagonalizable. If possible, find an invertible matrix $\mathbf{S}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{D}$. Do not use technology.

Problem 9. $\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **Problem 10.** $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **Problem 11.** $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Problem 12. For the matrix $\mathbf{A} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$, find formulas for the entries of $\mathbf{A}^t$, where *t* is a positive integer.

Also, find the vector $\mathbf{A}^t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

In Problems 13 and 14, find all the eigenvalues and “eigenvectors” of the given linear transformations.

Problem 13. $[T(f)](x) = f(2x)$ from P_2 to P_2. [Here P_2 is the space of all polynomials of degree ≤ 2. Is T diagonalizable? [Start by choosing a basis and finding the matrix of T relative to that basis.]

Problem 14. $[T(f)](x) = f(x-3)$ from P_2 to P_2. Is T diagonalizable?

In Problems 15 and 16, find all the complex eigenvalues of the matrices (including the real ones, of course). Do not use technology. Show all your work.

Problem 15. $\mathbf{A} = \begin{bmatrix} 3 & -5 \\ 2 & -3 \end{bmatrix}$ **Problem 16.** $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -7 & 3 \end{bmatrix}$

Problem 17. Suppose a 3×3 matrix $\mathbf{A}$ has the real eigenvalue 2 and two complex eigenvalues. Also suppose that $\det(\mathbf{A}) = 50$ and $\operatorname{tr}(\mathbf{A}) = 8$. Find the complex eigenvalues.

Problem 18. A real $n \times n$ matrix $\mathbf{A}$ is called a regular transition matrix if all entries of $\mathbf{A}$ are positive, and the entries in each column add up to 1. An example is $\mathbf{A} = \begin{bmatrix} 0.4 & 0.3 & 0.1 \\ 0.5 & 0.1 & 0.2 \\ 0.1 & 0.6 & 0.7 \end{bmatrix}$.

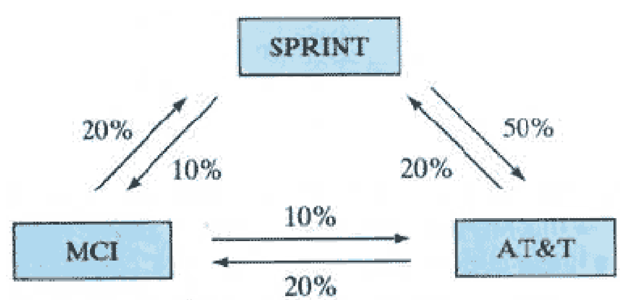
You may take the following properties of a regular transition matrix for granted:

- 1 is an eigenvalue of $\mathbf{A}$, with $\dim(E_1) = 1$.
- If λ is a complex eigenvalue of $\mathbf{A}$ other than 1, then $|\lambda| < 1$.

a. Consider a regular $n \times n$ transition matrix $\mathbf{A}$ and a vector $\mathbf{x}$ in $\mathbf{R}^n$ whose entries add up to 1. Show that the entries of $\mathbf{Ax}$ will also add up to 1.

b. Pick a regular transition matrix $\mathbf{A}$, and compute some powers of $\mathbf{A}$ (using technology): $\mathbf{A}^2, \ldots, \mathbf{A}^{10}, \ldots, \mathbf{A}^{100}, \ldots$. What do you observe? Explain your observation. Here, you may assume that there is a complex eigenbasis for $\mathbf{A}$.

Problem 19. Before the age of cell phones, most long-distance telephone service in the United States was provided by three companies: AT&T, MCI, and Sprint. The three companies were in fierce competition, offering discounts or even cash to those who switched. If the figures advertised by the companies were to be believed, people were switching their long-distance provider from one month to the next according to the following diagram:



For example, 20% of the people who use AT&T go to Sprint one month later.

a. We introduce the state vector

$$\mathbf{x}(t) = \begin{bmatrix} a(t) \\ m(t) \\ s(t) \end{bmatrix} \begin{matrix} \text{fraction using AT\&T} \\ \text{fraction using MCI} \\ \text{fraction using Sprint} \end{matrix}$$

Find the matrix $\mathbf{A}$ such that $\mathbf{x}(t+1) = \mathbf{A}[\mathbf{x}(t)]$, assuming that the customer base remains unchanged. Note that $\mathbf{A}$ is a regular transition matrix.

b. Which fraction of the customers will be with each company in the long term? Do you have to know the current market shares to answer this question? Use the power method introduced in Problem 18 (or a more analytic approach).

For additional practice:

Section 7.3:

27. Consider a 2×2 matrix **A**. Suppose that $\operatorname{tr}(\mathbf{A}) = 5$ and $\det(\mathbf{A}) = 6$. Find the eigenvalues of **A**.

33. Show that if matrix **A** is similar to **B**, then $\mathbf{A} - \lambda\mathbf{I}_n$ is similar to $\mathbf{B} - \lambda\mathbf{I}_n$, for all scalars λ.

35. Is matrix $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ similar to $\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$?

47. The color of snapdragons is determined by a pair of genes, which we designate by the letters A and a. The pair of genes is called the flower's *genotype*. Genotype AA produces red flowers, genotype Aa pink ones, and genotype aa white ones. A biologist undertakes a breeding program, starting with a large population of flowers of genotype AA. Each flower is fertilized with pollen from a plant of genotype Aa (taken from another population), and one offspring is produced. Since it is a matter of chance which of the genes a parent passes on, we expect half of the flowers in the next generation to be red (genotype AA) and the other half pink (genotype Aa). All the flowers in this generation are now fertilized with pollen from plants of genotype Aa (taken from another population), and so on.

   a. Find closed formulas for the fractions of red, pink, and white flowers in the t-th generation. We know that $r(0) = 1$ and $p(0) = w(0) = 0$, and we found that $r(1) = p(1) = ½$ and $w(1) = 0$.

   b. What is the proportion $r(t) : p(t) : w(t)$ in the long term?

Section 7.4:

In Exercises 4, 5, and 11, determine if **A** is diagonalizable. If possible, find an invertible **S** and a diagonal **D** such that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$. Do not use technology.

4. $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ 5. $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ 11. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

In Exercises 47 and 49, find all the eigenvalues and "eigenvectors" of the given linear transformations.

47. $[T(f)](x) = f(-x)$ from P_2 to P_2. Is T diagonalizable?

49. $[T(f)](x) = f(3x-1)$ from P_2 to P_2. Is T diagonalizable?

Section 7.5:

Find all complex eigenvalues of the matrices in Exercises 21 and 22 (including the real ones, of course). Do not use technology. Show all your work.

21. $\mathbf{A} = \begin{bmatrix} 11 & -15 \\ 6 & -7 \end{bmatrix}$ 22. $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -4 & 10 \end{bmatrix}$

27. Suppose a 3×3 matrix **A** has only two distinct eigenvalues. Suppose that $\operatorname{tr}(\mathbf{A}) = 1$ and $\det(\mathbf{A}) = 3$. Find the eigenvalues of **A** with their algebraic multiplicities.

29. Consider a matrix of the form $\mathbf{A} = \begin{bmatrix} 0 & a & b \\ c & 0 & 0 \\ 0 & d & 0 \end{bmatrix}$ where a, b, c, and d are positive real numbers. Suppose the matrix **A** has three distinct real eigenvalues. What can you say about the signs of the eigenvalues? (How many of them are positive, negative, zero?) Is the eigenvalue with the largest absolute value positive or negative?

36. In 1990, the population of the African country Benin was about 4.6 million people. Its composition by age was as follows:

Age Bracket	0-15	15-30	30-45	45-60	60-75	75-90
Percent of Population	46.6	25.7	14.7	8.4	3.8	0.8

We represent these data in a state vector whose components are the populations in the various age brackets, in millions:

$$\mathbf{x}(0) = 4.6\begin{bmatrix} 0.466 \\ 0.257 \\ 0.147 \\ 0.084 \\ 0.038 \\ 0.008 \end{bmatrix} \approx \begin{bmatrix} 2.14 \\ 1.18 \\ 0.68 \\ 0.39 \\ 0.17 \\ 0.04 \end{bmatrix}$$

We measure time in increments of 15 years, with $t = 0$ in 1990. For example, $\mathbf{x}(3)$ gives the age composition in the year 2035 (1990 + 3 · 15). If current age-dependent birth and death rates are extrapolated, we have the following model:

$$\mathbf{x}(t+1) = \begin{bmatrix} 1.1 & 1.6 & 0.6 & 0 & 0 & 0 \\ 0.82 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.89 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.81 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.53 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.29 & 0 \end{bmatrix}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t)$$

a. Explain the significance of all the entries in the matrix **A** in terms of population dynamics.

b. Find the eigenvalue of **A** with largest modulus and associated eigenvalue. (Use technology.) What is the significance of these quantities in terms of population dynamics? (For a summary on matrix techniques used in the study of age-structured populations, see Dmitrii O. Logofet, *Matrices and Graphs: Stability Problems in Mathematical Ecology*, Chapters 2 and 3, CRC Press, 1993.)

Chapter 7 True/False

1. The algebraic multiplicity of an eigenvalue cannot exceed its geometric multiplicity.

2. If an $n \times n$ matrix A is diagonalizable (over $\mathbb{R}$), then there must be a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

3. If the standard vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ are eigenvectors of an $n \times n$ matrix A, then A must be diagonal.

4. If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ must be an eigenvector of A^3 as well.

5. There exists a diagonalizable 5×5 matrix with only two distinct eigenvalues (over $\mathbb{C}$).

6. There exists a real 5×5 matrix without any real eigenvalues.

7. If 0 is an eigenvalue of a matrix A, then $\det A = 0$.

8. The eigenvalues of a 2×2 matrix A are the solutions of the equation $\lambda^2 - (\mathrm{tr}A)\lambda + (\det A) = 0$.

9. The eigenvalues of any triangular matrix are its diagonal entries.

10. The trace of any square matrix is the sum of its diagonal entries.

11. Any rotation-scaling matrix in $\mathbb{R}^{2\times 2}$ is diagonalizable over $\mathbb{C}$.

12. If A is a noninvertible $n \times n$ matrix, then the geometric multiplicity of eigenvalue 0 is $n - \mathrm{rank}(A)$.

13. If matrix A is diagonalizable, then its transpose A^T must be diagonalizable as well.

14. If A and B are two 3×3 matrices such that $\mathrm{tr}\, A = \mathrm{tr}\, B$ and $\det A = \det B$, then A and B must have the same eigenvalues.

15. If 1 is the only eigenvalue of an $n \times n$ matrix A, then A must be I_n.

16. If A and B are $n \times n$ matrices, if α is an eigenvalue of A, and if β is an eigenvalue of B, then $\alpha\beta$ must be an eigenvalue of AB.

17. If 3 is an eigenvalue of an $n \times n$ matrix A, then 9 must be an eigenvalue of A^2.

18. The matrix of any orthogonal projection onto a subspace V of $\mathbb{R}^n$ is diagonalizable.

19. If matrices A and B have the same eigenvalues (over $\mathbb{C}$), with the same algebraic multiplicities, then matrices A and B must have the same trace.

20. If a real matrix A has only the eigenvalues 1 and -1, then A must be orthogonal.

21. If an invertible matrix A is diagonalizable, then A^{-1} must be diagonalizable as well.

22. If $\det(A) = \det(A^T)$, then matrix A must be symmetric.

23. If matrix $A = \begin{bmatrix} 7 & a & b \\ 0 & 7 & c \\ 0 & 0 & 7 \end{bmatrix}$ is diagonalizable, then a, b, and c must all be zero.

24. If two $n \times n$ matrices A and B are diagonalizable, then $A + B$ must be diagonalizable as well.

25. All diagonalizable matrices are invertible.

26. If vector $\vec{v}$ is an eigenvector of both A and B, then $\vec{v}$ must be an eigenvector of $A + B$.

27. If matrix A^2 is diagonalizable, then matrix A must be diagonalizable as well.

28. The determinant of a matrix is the product of its eigenvalues (over $\mathbb{C}$), counted with their algebraic multiplicities.

29. All lower triangular matrices are diagonalizable (over $\mathbb{C}$).

30. If two $n \times n$ matrices A and B are diagonalizable, then AB must be diagonalizable as well.

31. If $\vec{u}$, $\vec{v}$, $\vec{w}$ are eigenvectors of a 4×4 matrix A, with associated eigenvalues 3, 7, and 11, respectively, then vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ must be linearly independent.

32. If a 4×4 matrix A is diagonalizable, then the matrix $A + 4I_4$ must be diagonalizable as well.

33. If an $n \times n$ matrix A is diagonalizable, then A must have n distinct eigenvalues.

34. If two 3×3 matrices A and B both have the eigenvalues 1, 2, and 3, then A must be similar to B.

35. If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ must be an eigenvector of A^T as well.

36. All invertible matrices are diagonalizable.

37. If $\vec{v}$ and $\vec{w}$ are linearly independent eigenvectors of matrix A, then $\vec{v} + \vec{w}$ must be an eigenvector of A as well.

38. If a 2×2 matrix R represents a reflection about a line L, then R must be diagonalizable.

39. If A is a 2×2 matrix such that $\operatorname{tr} A = 1$ and $\det A = -6$, then A must be diagonalizable.

40. If a matrix is diagonalizable, then the algebraic multiplicity of each of its eigenvalues λ must equal the geometric multiplicity of λ.

41. All orthogonal matrices are diagonalizable (over $\mathbb{R}$).

42. If A is an $n \times n$ matrix and λ is an eigenvalue of the block matrix $M = \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$, then λ must be an eigenvalue of matrix A.

43. If two matrices A and B have the same characteristic polynomials, then they must be similar.

44. If A is a diagonalizable 4×4 matrix with $A^4 = 0$, then A must be the zero matrix.

45. If an $n \times n$ matrix A is diagonalizable (over $\mathbb{R}$), then every vector $\vec{v}$ in $\mathbb{R}^n$ can be expressed as a sum of eigenvectors of A.

46. If vector $\vec{v}$ is an eigenvector of both A and B, then $\vec{v}$ is an eigenvector of AB.

47. Similar matrices have the same characteristic polynomials.

48. If a matrix A has k distinct eigenvalues, then $\operatorname{rank}(A) \geq k$.

49. If the rank of a square matrix A is 1, then all the nonzero vectors in the image of A are eigenvectors of A.

50. If the rank of an $n \times n$ matrix A is 1, then A must be diagonalizable.

51. If A is a 4×4 matrix with $A^4 = 0$, then 0 is the only eigenvalue of A.

52. If two $n \times n$ matrices A and B are both diagonalizable, then they must commute.

53. If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ must be in the kernel of A or in the image of A.

54. All symmetric 2×2 matrices are diagonalizable (over $\mathbb{R}$).

55. If A is a 2×2 matrix with eigenvalues 3 and 4 and if $\vec{u}$ is a unit eigenvector of A, then the length of vector $A\vec{u}$ cannot exceed 4.

56. If $\vec{u}$ is a nonzero vector in $\mathbb{R}^n$, then $\vec{u}$ must be an eigenvector of matrix $\vec{u}\vec{u}^T$.

57. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is an eigenbasis for both A and B, then matrices A and B must commute.

58. If $\vec{v}$ is an eigenvector of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\vec{v}$ must be an eigenvector of its classical adjoint $\operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ as well.