

## Integration on Surfaces - Toolkits

It is often necessary to measure the aggregate amount of *something* on a surface. Examples include the total area of a surface, total charge for a given charge density function, total population of people on a planet, or total flux (flow) of a vector field through a surface. A general surface integral of a function  $g(x, y, z)$  over a surface  $S$  is denoted by  $\iint_S g \, dS$  where, as always, the integral represents the limit of Riemann Sum.

The main tools for calculating such integrals are (a) parameterization of the surface (or, equivalently, finding two coordinates defined on the surface that provide a “mesh” for the surface), and (b) an expression for the “element of surface area”  $dS$  defined by the parameterization or coordinates. For flux integrals, it’s also handy to have an expression for a unit normal vector  $\mathbf{n}$  for the surface. Though it’s possible to develop an all-purpose method for surface integrals, it is generally easier and more geometrically clear to focus on several special cases before developing the general method, specifically the cases of cylinders, spheres, and graphs. We’ll refer to the methods for handling these cases as **toolkits**.

### Cylinder

Cartesian equation:  $x^2 + y^2 = R^2$

Parameterization:  $\begin{cases} x = R \cos \theta \\ y = R \sin \theta \\ z = z \end{cases}$ . The parameter  $\theta$  allows movement around the cylinder with

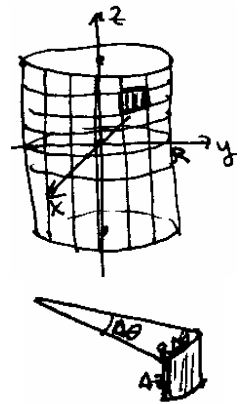
$0 \leq \theta \leq 2\pi$ , and the parameter  $z$  (which does double-duty as both a coordinate and a parameter) allows movement up and down the cylinder.

Surface area element: If we vary  $\theta$  by an amount  $\Delta\theta$ , we move a distance  $R\Delta\theta$  around the cylinder, and if we also move an amount  $\Delta z$  to span a “patch” of the surface, the area of this patch will be  $\Delta S = (R\Delta z)(\Delta\theta)$ . Within a Riemann Sum expression as the limit of the mesh tends to zero, this yields the surface area element  $dS = Rdz d\theta$  for a cylinder.

Unit normal vector: We can use gradient methods or observation to see that at any point  $(x, y, z)$  on the

cylinder, the outward unit normal vector to the surface is  $\mathbf{n} = \frac{\langle x, y, 0 \rangle}{R} = \langle \cos \theta, \sin \theta, 0 \rangle$ .

This method can be modified as necessary for cylinders around the  $x$ -axis or  $y$ -axis.



### Sphere

Cartesian equation:  $x^2 + y^2 + z^2 = R^2$

Parameterization:  $\begin{cases} x = R \cos \theta \sin \phi \\ y = R \sin \theta \sin \phi \\ z = R \cos \phi \end{cases}$ . The parameter  $\theta$  (longitude or azimuth) allows

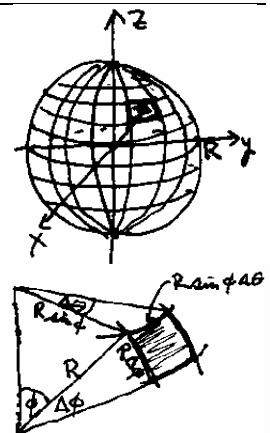
movement around the sphere with  $0 \leq \theta \leq 2\pi$ , and the parameter  $\phi$  (the inclination, related to latitude) allows movement up and down the sphere with  $0 \leq \phi \leq \pi$ .

Surface area element: If we vary  $\phi$  by an amount  $\Delta\phi$ , we move a distance  $R\Delta\phi$  along a longitude, and if we also move an amount  $\Delta\theta$  along a latitude (at a radius from the  $z$ -axis of  $r = R \sin \phi$ ), we will move a distance  $R \sin \phi \Delta\theta$  along a latitude. Together, these will span a “patch” of the surface with area approximately  $\Delta S \cong (R \sin \phi \Delta\theta)(R\Delta\phi) = R^2 \sin \phi \Delta\theta \Delta\phi$ . Within a Riemann Sum expression as the limit of the mesh tends to zero, this yields the surface area element

$dS = R^2 \sin \phi \, d\phi \, d\theta$  for a sphere.

Unit normal vector: We can use gradient methods or observation to see that at any point  $(x, y, z)$  on the sphere,

the outward unit normal vector to the surface is  $\mathbf{n} = \frac{\langle x, y, z \rangle}{R} = \langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle$ .

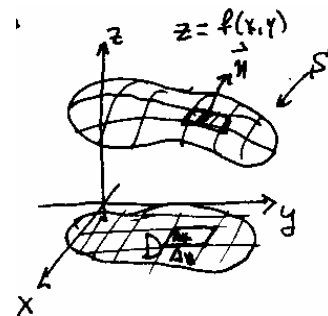


**Graph**  $z = f(x, y)$

Cartesian equation:  $z = f(x, y)$

Parameterization:  $\begin{cases} x = x \\ y = y \\ z = f(x, y) \end{cases}$  or  $\mathbf{r}(x, y) = \langle x, y, f(x, y) \rangle$ . The variables  $x$  and  $y$  here do

double-duty as both parameters and coordinates. They vary in the  $xy$ -plane over the domain of the function that describes this graph surface.



Surface area element: If we vary  $x$  by an amount  $\Delta x$ , we move an approximate vector displacement along a cross-section of the graph of  $\frac{\partial \mathbf{r}}{\partial x} \Delta x = \langle 1, 0, f_x \rangle \Delta x$ , and if we also vary  $y$  by an amount  $\Delta y$ , we move an approximate vector displacement along a cross-section of the graph of  $\frac{\partial \mathbf{r}}{\partial y} \Delta y = \langle 0, 1, f_y \rangle \Delta y$ . Together, these two displacements will span a “patch” of the surface and we can use the cross-product to determine its approximate area. Specifically,

$$\left( \frac{\partial \mathbf{r}}{\partial x} \Delta x \right) \times \left( \frac{\partial \mathbf{r}}{\partial y} \Delta y \right) = \left( \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right) \Delta x \Delta y = \langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle \Delta x \Delta y = \langle -f_x, -f_y, 1 \rangle \Delta x \Delta y.$$

This is a vector with magnitude approximately equal to the area of a “patch” on the graph surface and direction normal to the surface (actually in the upward normal direction). The magnitude is

$\Delta S \cong \sqrt{1 + f_x^2 + f_y^2} \Delta x \Delta y$ . Within a Riemann Sum expression as the limit of the mesh tends to zero, this

yields the surface area element  $dS \cong \sqrt{1 + f_x^2 + f_y^2} dx dy$  for a graph surface. We can also make use of the

“vector element of surface area”  $d\mathbf{S} = \mathbf{n} dS = \langle -f_x, -f_y, 1 \rangle dx dy$  where  $\mathbf{n}$  denotes the (upward) unit normal vector to the graph surface.

Unit normal vector: From the above calculation, we see that a unit (upward) normal vector to a graph surface is

$$\mathbf{n} = \frac{\langle -f_x, -f_y, 1 \rangle}{\sqrt{1 + f_x^2 + f_y^2}}. \text{ In class we also showed that } dS = \frac{dx dy}{\mathbf{n} \cdot \mathbf{k}} = \sqrt{1 + f_x^2 + f_y^2} dx dy, \text{ and this can be easily}$$

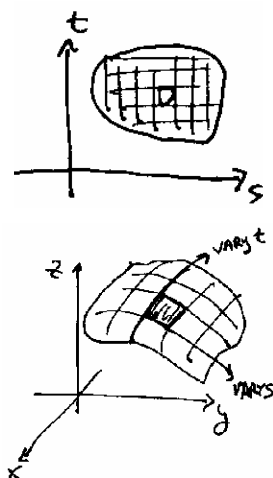
adapted to the case where  $x = f(y, z)$  or  $y = f(x, z)$ .

### General method for any parameterized surface

Parameterization: Suppose  $S$  is a surface parameterized by a vector-valued function

$$\mathbf{r}(s, t) = \langle x(s, t), y(s, t), z(s, t) \rangle \text{ where the parameters } s \text{ and } t \text{ vary over some domain in}$$

the parameter space. The only requirement is that the curves in the surface produced by varying one parameter at a time provide a mesh on the surfaces, i.e. these curves should “intersect cleanly” and produce a patchwork of small cells on the surface that can be used to build Riemann Sums on the surface.



Surface area element: If we vary  $s$  by an amount  $\Delta s$ , we move an approximate vector displacement along a cross-section of the graph of  $\frac{\partial \mathbf{r}}{\partial s} \Delta s$ , and if we also vary  $t$  by an amount  $\Delta t$ , we move an approximate vector displacement along a cross-section of the graph of  $\frac{\partial \mathbf{r}}{\partial t} \Delta t$ .

These two displacements will span a “patch” of the surface and the cross-product can be used to determine its approximate area. Specifically,

$$\left(\frac{\partial \mathbf{r}}{\partial s} \Delta s\right) \times \left(\frac{\partial \mathbf{r}}{\partial t} \Delta t\right) = \left(\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right) \Delta s \Delta t.$$

This is a vector with magnitude approximately equal to the area of a “patch” on the graph surface and direction normal to the surface (actually in the upward normal direction). The magnitude is

$\Delta S \equiv \left\| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right\| \Delta s \Delta t$ . Within a Riemann Sum expression as the limit of the mesh tends to zero, this yields

the surface area element  $dS = \left\| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right\| ds dt$  for a general parameterized surface. We can also make use of

the “vector element of surface area”  $d\mathbf{S} = \mathbf{n} dS = \left( \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right) ds dt$  where  $\mathbf{n}$  denotes the unit normal vector to

the surface, oriented in a manner consistent with the cross-product.

Unit normal vector: From the above calculation, we see that a unit normal vector to the surface with orientation

consistent with the cross product is  $\mathbf{n} = \frac{\left( \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right)}{\left\| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right\|}$ .

### Examples:

(1) Surface area of a sphere  $S$  of radius  $R$ :

Area( $S$ ) =  $\iint_S dS = \int_0^{2\pi} \int_0^\pi R^2 \sin \phi d\phi d\theta$ . The inner integral gives  $-R^2 \cos \phi \Big|_{\phi=0}^{\phi=\pi} = -R^2(-1-1) = 2R^2$ , and the outer integral gives  $(2R^2)(2\pi) = 4\pi R^2$ .

(2) The flux of the vector field  $\mathbf{F} = \langle 3x, -y, z^2 \rangle$  outward through a sphere of radius 2 centered at the origin:

Flux =  $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS$ . We use the unit outward normal vector  $\mathbf{n} = \frac{\langle x, y, z \rangle}{2}$ , so

$\mathbf{F} \cdot \mathbf{n} = \langle 3x, -y, z^2 \rangle \cdot \frac{\langle x, y, z \rangle}{2} = \frac{3x^2 - y^2 + z^3}{2}$ . Therefore, Flux =  $\iint_S \left( \frac{3x^2 - y^2 + z^3}{2} \right) dS$ . If we substitute the

parameterization  $\begin{cases} x = 2 \cos \theta \sin \phi \\ y = 2 \sin \theta \sin \phi \\ z = 2 \cos \phi \end{cases}$  and the area element  $dS = 4 \sin \phi d\phi d\theta$ , we get

$$\text{Flux} = \int_0^{2\pi} \int_0^\pi (24 \cos^2 \theta \sin^3 \phi - 8 \sin^2 \theta \sin^3 \phi + 16 \cos^3 \phi \sin \phi) d\phi d\theta.$$

This may not be the simplest integral, but it’s quite doable. The inner integral gives

$$\begin{aligned} & \int_0^\pi (24 \cos^2 \theta (1 - \cos^2 \phi) \sin \phi - 8 \sin^2 \theta (1 - \cos^2 \phi) \sin \phi + 16 \cos^3 \phi \sin \phi) d\phi \\ &= [24 \cos^2 \theta - 8 \sin^2 \theta] \left[ -\cos \phi + \frac{\cos^3 \phi}{3} \right]_0^\pi - 4 [\cos^4 \phi]_0^\pi = [24 \cos^2 \theta - 8 \sin^2 \theta] \left[ \frac{4}{3} \right] - 4[0] \\ &= \frac{4}{3} \left[ 24 \left( \frac{1 + \cos 2\theta}{2} \right) - 8 \left( \frac{1 - \cos 2\theta}{2} \right) \right] = \frac{4}{3} (8 + 16 \cos 2\theta) \end{aligned}$$

The outer integral is then  $\frac{4}{3} \int_0^{2\pi} (8 + 16 \cos 2\theta) d\theta = \frac{32}{3} \cdot 2\pi = \frac{64\pi}{3}$ .

This integral can also be (more simply) done using the **Divergence Theorem**. We calculate  $\text{div } \mathbf{F} = 3 - 1 + 2z = 2 + 2z$ , and

$$\text{Flux} = \oiint_{S=\text{Bnd}(B)} \mathbf{F} \cdot d\mathbf{S} = \iiint_B (\text{div } \mathbf{F}) dV = \iiint_B (2 + 2z) dV = \iiint_B 2 dV = 2 \cdot \text{Vol}(B) = 2 \cdot \left( \frac{4}{3} \pi \cdot 2^3 \right) = \frac{64\pi}{3}.$$