Math E-21a – Fall 2018 – HW #12 problems

Problems due Thurs, Dec 6 in class or no later than Dec 8 via Canvas:

Section 13.4:

4. Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.
$\oint_C x\,dx + y\,dy$, C consists of the line segments from (0,1) to (0,0) and from (0,0) to (1,0) and the parabola $y = 1 - x^2$ from (1,0) to (0,1).

6. Use Green's Theorem to evaluate the line integral $\oint_C \cos y\,dx + x^2 \sin y\,dy$ where C is the rectangle with vertices $(0,0)$, $(5,0)$, $(5,2)$, and $(0,2)$ traversed counterclockwise.

10. Use Green's Theorem to evaluate the line integral $\oint_C \sin y\,dx + x\cos y\,dy$ along the positively oriented curve C that is the ellipse $x^2 + xy + y^2 = 1$.

12. Use Green's Theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \left\langle y^2\cos x,\, x^2 + 2y\sin x \right\rangle$ and C is the triangle from $(0,0)$ to $(2,6)$ to $(2,0)$ to $(0,0)$.

18. A particle starts at the point $(-2,0)$, moves along the x-axis to $(2,0)$, and then along the semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x,y) = \left\langle x,\, x^3 + 3xy^2 \right\rangle$.

22. Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem to prove that the coordinates of the centroid $(\bar{x},\bar{y})$ of D are $\bar{x} = \frac{1}{2A}\oint_C x^2 dy \qquad \bar{y} = -\frac{1}{2A}\oint_C y^2 dx$
where A is the area of D (and where the curve C is traversed in the counterclockwise sense).

24. Use Exercise 22 to find the centroid of the triangle with vertices $(0,0)$, $(a,0)$, and (a,b).

Section 13.5:

2. Find the curl and the divergence of the vector field $\mathbf{F}(x,y,z) = x^2yz\,\mathbf{i} + xy^2z\,\mathbf{j} + xyz^2\,\mathbf{k}$.

14. Determine whether or not the vector field $\mathbf{F}(x,y,z) = xyz^2\,\mathbf{i} + x^2yz^2\,\mathbf{j} + x^2y^2z\,\mathbf{k}$ is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

18. Determine whether or not the vector field $\mathbf{F}(x,y,z) = y\cos xy\,\mathbf{i} + x\cos xy\,\mathbf{j} - \sin z\,\mathbf{k}$ is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

20. Is there a vector field $\mathbf{G}$ on $\mathbf{R}^3$ such that $\operatorname{curl}\mathbf{G} = \left\langle xyz, -y^2z, yz^2 \right\rangle$? Explain.
[*Hint*: Use one of the two identities $\operatorname{curl}(\nabla f) = \mathbf{0}$ and $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$ for any twice differentiable function f or any vector field $\mathbf{F}$ with components that are at least twice differentiable. These follow from Clairaut's Theorem on equality of mixed partial derivatives.]

26. Prove the identity $\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + (\nabla f)\times\mathbf{F}$, assuming the appropriate partial derivatives exist and are continuous.

38. Maxwell's equations relating the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ as they vary with time in a region containing no charge and no current can be stated as follows:

$$\operatorname{div}\mathbf{E} = 0, \quad \operatorname{div}\mathbf{H} = 0, \quad \operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \quad \operatorname{curl}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

where c is the speed of light. Use these equations to prove the following:

(a) $\nabla\times(\nabla\times\mathbf{E}) = -\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}$ (b) $\nabla\times(\nabla\times\mathbf{H}) = -\frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}$ (c) $\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}$ (d) $\nabla^2\mathbf{H} = \frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}$

Hint: For (c) and (d), use the identity $\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla^2\mathbf{F}$ (from Exercise 29 – see below).

Note: In Exercises 13.5/29 and 13.5/38, reference is made to the Laplacian $\nabla^2\mathbf{F}$ where **F** is a vector field. We generally only refer to the Laplacian of a function, i.e. $\nabla^2 f = \text{div}(\text{grad } f) = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$, but we can extend this to vector fields by defining $\nabla^2\mathbf{F}$ to be the vector field whose components are the Laplacians of its respective component functions.

Extra-credit Challenge Problem [submit separately from the rest of your homework]:
Let Ω be a convex region in $\mathbf{R}^2$ and let L be a line segment of length I that connects points on the boundary of Ω. As we move one end of L around the boundary, the other end will also move about this boundary, and the midpoint of L will trace out a curve within Ω that bounds a (smaller) region Γ. Using the corollary to Green's Theorem for finding area, find an expression that relates the area of Γ to the area of Ω in terms of the length I of the line segment. [You might start with some simple regions, but you must show this generally.]

For additional practice:

Section 13.4:

3. Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

$$\oint_C xy\,dx + x^2y^3\,dy, \quad C \text{ is the triangle with vertices } (0,0), (1,0), \text{ and } (1,2).$$

13. Use Green's Theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \langle e^x + x^2y, e^y - xy^2 \rangle$ and C is the circle $x^2 + y^2 = 25$ oriented clockwise.

Section 13.5:

1. Find the curl and the divergence of the vector field $\mathbf{F}(x,y,z) = xyz\,\mathbf{i} - x^2y\,\mathbf{k}$.

15. Determine whether or not the vector field $\mathbf{F}(x,y,z) = 2xy\,\mathbf{i} + (x^2 + 2yz)\,\mathbf{j} + y^2\,\mathbf{k}$ is conservative. If it is conservative, find a function *f* such that $\mathbf{F} = \nabla f$.

16. Determine whether or not the vector field $\mathbf{F}(x,y,z) = e^z\,\mathbf{i} + \mathbf{j} + xe^z\,\mathbf{k}$ is conservative. If it is conservative, find a function *f* such that $\mathbf{F} = \nabla f$.

25. Prove the identity $\text{div}(f\mathbf{F}) = f\,\text{div}\,\mathbf{F} + (\nabla f)\cdot\mathbf{F}$, assuming the appropriate partial derivatives exist and are continuous.

29. Prove the identity $\text{curl}(\text{curl}\,\mathbf{F}) = \text{grad}(\text{div}\,\mathbf{F}) - \nabla^2\mathbf{F}$, assuming the appropriate partial derivatives exist and are continuous. [The quantity $\nabla^2 f = \text{div}(\text{grad } f) = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$ is known as the *Laplacian* of *f*.]

37. This exercise demonstrates a connection between the curl vector and rotations. Let *B* be a rigid body rotating about the *z*-axis. The rotation can be described by the vector $\mathbf{w} = \omega\,\mathbf{k}$, where ω is the angular speed of *B*, that is, the tangential speed of any point *P* in *B* divided by the distance *d* from the axis of rotation.

(a) By considering the angle θ in the figure, show that the velocity vector field of *B* is given by $\mathbf{v} = \mathbf{w}\times\mathbf{r}$.

(b) Show that $\mathbf{v} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$.

(c) Show that $\text{curl}\,\mathbf{v} = 2\mathbf{w}$.