Problem Set #6 – Concourse 18.03 – Spring 2019
(due Thurs, Apr 11)

Problem 1: Find the smallest period for each of the following:

a) $\sin(\pi t/3)$ b) $|\sin t|$ c) $\cos^2(3t)$

Problem 2: a) Using integral formulas for the Fourier coefficients, find the Fourier series of the function $f(t)$ of period 2π which is given over the interval $-\pi < t \le \pi$ by

$$f(t) = \begin{cases} 0 & -\pi < t \le 0 \\ 1 & 0 < t \le \pi \end{cases}$$

b) Find the Fourier series of this same function $f(t)$ – but this time use the known Fourier series for $sq(t)$ [= the standard square wave] to derive the Fourier series. The square wave $sq(t)$ is the odd function of period 2π such that $sq(t) = 1$ for $0 < t < \pi$ and $sq(\pi) = 0$. We calculated its Fourier series in class.

Problem 3: Find the Fourier series of the function $f(t)$ of period 2π given by $f(t) = |t|$ on $(-\pi, \pi)$ by integrating the Fourier series of the derivative $f'(t)$.

Problem 4: Find a periodic solution as a Fourier series to $\ddot{x} + 3x = F(t)$, where $F(t) = 2t$ on $(0, \pi)$, $F(t)$ is odd, and of period 2π.

Problem 5: Find a periodic solution as a Fourier series to $\ddot{x} + 4x = f(t)$, where $f(t)$ of period 2π is given by $f(t) = |t|$ on $(-\pi, \pi)$.

Problem 6: For each spring-mass system, find whether pure resonance occurs, without actually calculating the solution.

a) $2\ddot{x} + 10x = F(t)$; $F(t) = 1$ on $(0,1)$, $F(t)$ is odd, and of period 2;

b) $\ddot{x} + 4\pi^2 x = F(t)$; $F(t) = 2t$ on $(0,1)$, $F(t)$ is odd, and of period 2;

c) $\ddot{x} + 9x = F(t)$; $F(t) = 1$ on $(0, \pi)$, $F(t)$ is odd, and of period 2π

Problem 7: [Fourier Series] This problem will use the Mathlet [**Fourier Coefficients**]. When the applet opens you are presented with a series of sliders labeled b_n. By pressing the [**Formula**] radio button you can see that they are coefficients of sines in a Fourier series made up entirely of sine functions. If you press the [**Cosine**] radio button you'll see a_n's, coefficients of cosines. Move one of the slider handles: a cosine or sine curve appears and changes amplitude. Release it at some value and move another one. The white curves shows the new sinusoid, and the yellow curve shows the sum of the two. By moving more sliders you can build up more complicated sums and more complicated functions.

Now select the Target [**B**]. Is it an even function or an odd function? Based on this, decide whether to approximate it using sines or cosines. Select one or the other appropriately (using [**All terms**]) and do the best you can by eyeballing the result to get the best approximation you can to the green target curve. Does it appear that only even terms are needed? Only odd term? or both? The [**Odd terms**] and [**Even terms**] buttons allow you to choose just the even terms or the odd terms, and gives you more of the one you select. If it seems that just the even or odd terms will be useful, explain (in words) why.

(a) Write down these values of the coefficients.

(b) The target function [**B**] is the periodic function with period 2π which is given by $f(t)=\frac{\pi}{4}$ for $-\frac{\pi}{2}<t<+\frac{\pi}{2}$; $f(t)=-\frac{\pi}{4}$ for $\frac{\pi}{2}<t<+\frac{3\pi}{2}$; and $f(\pm\frac{\pi}{4})=0$. Compute the Fourier coefficients for this function, using the integral formulas for them, and compare with your answers from (a). Reset the sliders to the computed values and see if it looks like a better fit.

(c) Now set the sliders to some random set of values. Still with target function [**B**] displayed, select the [**Distance**] button. A number appears at the upper right corner of the screen. This is the "root mean square" distance from the target function to the selected finite Fourier sum. It is a measure of goodness of fit. Instead of eyeballing the fit as before, start from the bottom and successively adjust the sliders to minimize the distance. Write down the optimal values of the coefficients. Compare with the computed values.

Lessons: **(1)** The Fourier coefficients are the coefficients resulting in the best possible fit, and **(2)** the process of optimizing one coefficient is independent of the process of optimizing any of the others. (This is "orthogonality.")

Problem 8: [Fourier Series]

(a) Find the Fourier series for $2\sin(t-\frac{\pi}{3})$ (Hint: A function of period 2π has just one expression as a linear combination of $\cos(mt)$'s and $\sin(nt)$'s.)

The square wave $sq(t)$ is the odd function of period 2π such that $sq(t)=1$ for $0<t<\pi$ and $sq(\pi)=0$. In class we calculated its Fourier series.

(b) $sq(t)$ has minimal period 2π, but it is also a function of period 4π. Use the integral formulas for the Fourier coefficients to calculate its Fourier series, regarded as a function of period 4π. Comment on the relationship between your answer and the Fourier series for $sq(t)$.

Use the Fourier series for $sq(t)$, along with calculus and algebraic manipulations, to compute the Fourier series of each of the following functions without evaluating any of the integrals for the Fourier coefficients. In each case, sketch a graph of the function, as well, and give the minimal period.

(c) $sq(t-\frac{\pi}{4})$.

(d) $1+2sq(2\pi t)$.

(e) The $f(t)$ of [**B**] in the [**Fourier Coefficients**] Applet explored in Part II, Problem 1 above.

(f) The periodic function $g(t)$ with period 2π such that $g(t)=t$ for $-\frac{\pi}{2}\le t\le+\frac{\pi}{2}$ and $g(t)=\pi-t$ for $\frac{\pi}{2}\le t\le+\frac{3\pi}{2}$. (Hint: what is $g'(t)$ in terms of $sq(t)$?)