MIT Concourse 18.03 – Spring 2019 – Problem Set #3
(due Wed, Mar 6 in 16-137)

Problem 1: Change to polar form: a) $-1+i$ b) $\sqrt{3}-i$

Problem 2: Express $\frac{1-i}{1+i}$ in the form $a+bi$ via two methods: one using the Cartesian form throughout, and one changing numerator and denominator to polar form. Show the two answers agree.

Problem 3: Calculate each of the following two ways: first by using the binomial theorem and second by changing to polar form and using DeMoivre's formula: $(\cos x + i\sin x)^n = \cos(nx) + i\sin(nx)$.
[Note: This is a simple corollary of Euler's formula $e^{ix} = \cos x + i\sin x$ and the exponential law $(e^{ix})^n = e^{inx}$.]
a) $(1-i)^4$ b) $(1+i\sqrt{3})^3$

Problem 4: [Complex numbers, roots of unity, complex exponentials]
a) Find the complex roots of the following equations: $z^4+4=0$; $z^2+2z+2=0$.
b) Find all complex numbers $z=a+bi$ such that $e^z=-2$.
c) Find an expression for $\cos(4t)$ in terms of sums of powers of $\sin t$ and $\cos t$ by using $(e^{it})^4 = e^{4it}$ and Euler's formula.

Problem 5: Write each of the following functions $f(t)$ in the form $A\cos(\omega t - \phi)$. In each case, begin by drawing a right triangle.
a) $2\cos(3t)+2\sin(3t)$ b) $\sqrt{3}\cos(\pi t)-\sin(\pi t)$ c) $\cos(t-\frac{\pi}{8})+\sin(t-\frac{\pi}{8})$

Problem 6: Find $\int e^{2x}\sin x\,dx$ by using complex exponentials.

Problem 7: a) Find a solution of $\dot{x}+2x=e^{3t}$ of the form Be^{3t}. Then find the general solution.
b) Now do the same for the complex-valued differential equation $\dot{z}+2z=e^{3it}$.
c) Use the result of part (b) to find the general solution for both the ODE $\dot{x}+2x=\cos 3t$ and the ODE $\dot{y}+2y=\sin 3t$. Express your particular solutions in two forms: (i) sum of trig functions, and (ii) amplitude-phase form.

Problem 8: For each of the following autonomous equations $\frac{dx}{dt}=f(x)$, obtain a qualitative picture of the solutions as follows:
i) Draw horizontally the axis of the dependent variable *x*, indicating the critical points of the equation; put arrows on the axis indicating the direction of motion between the critical points and label each critical point as stable, unstable, or semi-stable. Indicate where this information comes from by including in the same picture the graph of $f(x)$, drawn with dashed lines.
ii) Use the information in the first picture to make a second picture showing the *tx*-plane, with a set of typical solutions to the ODE. The sketch should show the main qualitative features (e.g., the constant solutions, asymptotic behavior of the non-constant solutions).

a) $x' = x^2+2x$

b) $x' = -(x-1)^2$

c) $x' = 2x-x^2$

d) $x' = (2-x)^3$

Problem 9: Consider the differential equation $\frac{dx}{dt} + 2x = 1$.

a) Find the general solution three ways:

i) by separation of variables,

ii) by use of an integrating factor, and

iii) by finding a particular solution to the inhomogeneous linear ODE using undetermined coefficients and an enlightened guess for a particular solution, and then adding in a transient (homogeneous solution).

b) This equation is also autonomous. Sketch its phase line and some solutions (including the equilibrium solution). Is the equilibrium stable, unstable, or neither?

Problem 10: [Linear vs Nonlinear]

Consider the nonlinear autonomous differential equation $\dot{y} = (1-y)y - \frac{3}{16}$, and let y_0 be the stable critical point. Write $u = y - y_0$ for the population excess over equilibrium (so $u < 0$ if the population is less than the equilibrium value).

a) Rewrite the differential equation as a differential equation for *u*. Check that the new equation is again autonomous and that $u = 0$ is a critical point for it.

b) For small *u* we can neglect higher powers of *u* (such as u^2). This process is “linearization near equilibrium.” What is the linearized equation near $u = 0$? What is the general solution of this linear autonomous equation?

c) At least in this case, when solutions of the original autonomous equation get near equilibrium, they are well modeled by solutions of the linearization. Give an approximation of *y* near equilibrium. Use it to answer this question: if $y(10) - y_0 = b$, estimate $y(11)$ and $y(12)$.

d) Suppose that the linear equation $\dot{x} + p(t)x = q(t)$ is autonomous. What can you say about $p(t)$ and $q(t)$?