Problem Set #10 – Concourse 18.03 – Spring 2019

(we will go over some or all of these in class, but these are not to be turned in)

Note, however, that you will be responsible on the Final Exam for phase-plane analysis of nonlinear systems.

Problem 1. (a) Give the general solution of the system $\frac{d\mathbf{x}}{dt}=\begin{bmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{bmatrix}\mathbf{x}$ with initial conditions $\mathbf{x}(0)$ in terms of the evolution matrix for the given matrix.

Problem 2. a) Find the general solution for the following system of differential equations:

$$\left\{\begin{aligned}\frac{dx_1}{dt}&=2x_1-4x_4+3x_5\\ \frac{dx_2}{dt}&=2x_2-2x_3+2x_4\\ \frac{dx_3}{dt}&=x_2-x_4\\ \frac{dx_4}{dt}&=-x_4\\ \frac{dx_5}{dt}&=-3x_4+2x_5\end{aligned}\right\}$$

b) Find the solution in the case where $\mathbf{x}(0) = (5, 4, 3, 2, 1)$.

Problem 3. The interaction of two species of animals is modeled by

$$\left\{\begin{aligned}\frac{dx}{dt}&=x(2-x+y)\\ \frac{dy}{dt}&=y(4-x-y)\end{aligned}\right\} \text{ for } x\geq 0 \text{ and } y\geq 0.$$

a) Sketch a phase portrait for this system. Make sure that your sketch clearly shows the nullclines and the equilibria.

b) There is one equilibrium point (a,b) with $a>0$ and $b>0$. Find the Jacobian matrix **J** of the system at that point.

c) Determine the stability of the equilibrium point (a,b) discussed in part (b).

Problem 4. Consider the system:

$$\left\{\begin{aligned}\frac{dx}{dt}&=x(1-x+ky-k)\\ \frac{dy}{dt}&=y(1-y+kx-k)\end{aligned}\right\} \text{ where } k \text{ is a constant different from 1 and } -1.$$

a) The system above has exactly one equilibrium point (a,b) in the first quadrant with $a>0$ and $b>0$. Find this equilibrium point.

b) Find the Jacobian matrix at the equilibrium point.

c) Determine the stability of the equilibrium point. Your answer may depend on the constant k.

Problem 5. The dynamics of a frictionless pendulum of length L are given by the system

$$\begin{cases} \dfrac{d\alpha}{dt} = \omega \\ \dfrac{d\omega}{dt} = -\dfrac{g}{L}\sin\alpha \end{cases}$$

where α is the angle the rod of the pendulum makes with the vertical line, $\omega = \dfrac{d\alpha}{dt}$ is the angular velocity, and g is the gravitational constant.

a) Sketch a phase portrait for this system. Think about the trajectories in terms of the motion of a frictionless pendulum.

b) Find the Jacobian matrix at all equilibrium point, and compute the eigenvalues. What does the answer tell you about the stability of the equilibria?

Problem 6. Consider the system

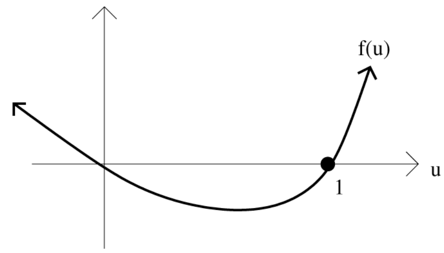
$$\begin{cases} \dfrac{dx}{dt} = x^2 + y^2 - 1 \\ \dfrac{dy}{dt} = xy \end{cases}.$$

Sketch a phase plane for this system. Make sure that your sketch clearly shows the nullclines and the equilibria. Which equilibria are stable?

Problem 7. In an article on insect dispersion we found the differential equation

$$\frac{d^2u}{dt^2} = f(u) - \frac{du}{dt}$$

where $f(u)$ is the function graphed below:



We can introduce the auxiliary function $v = \dfrac{du}{dt}$ and write the second order differential equation above as a system:

$$\begin{cases} \dfrac{du}{dt} = v \\ \dfrac{dv}{dt} = f(u) - v \end{cases}$$

Sketch the phase portrait of this system, clearly identifying the nullclines and the equilibria. Use Jacobian matrices to determine the stability of the equilibria.