MIT Concourse 18.03 – Spring 2019 – Problem Set #1
(due Thurs, Feb 14, 2019 at 4:00pm in 16-137)

Problem 1: Sketch the slope field for the ordinary differential equation (ODE) given below, and solve it, i.e. express $y = y(x)$ for the given initial condition by separation of variables:

$$\frac{dy}{dx} = 2 - y, \quad y(0) = 0$$

Problem 2: The amount of a certain medicine in the bloodstream decays exponentially with a half-life of 5 hours. In order to keep a patient safe during a one-hour procedure, there needs to be at least 50 mg of medicine per kg of body weight. How much medicine should be administered to a 60kg patient at the start of the procedure?

Problem 3: Early one morning it starts to snow steadily, i.e. at a constant rate of accumulation. At 7AM a snowplow sets off to clear the road. By 8AM, it has gone 2 miles. It takes an additional 2 hours for the plow to go another 2 miles. Let $t = 0$ when it begins to snow, let x denote the distance traveled by the plow at time t. Assuming the snowplow clears snow at a constant rate in cubic meters/hour:

a) Find the differential equation modeling the value of x.

b) When did it start snowing?

Problem 4: A tank holds 100 liters of water which contains 25 grams of salt initially. Pure water then flows into the tank, and salt water flows out of the tank, both at 5 liters/minute. The mixture is kept uniform at all times by stirring.

a) Write down the differential equation with initial condition for this situation.

b) How long will it take until only 1 gram of salt remains in the tank?

Problem 5: Water flows into and out of a 100,000 liter (l) reservoir at a constant rate of 10 l/min. The reservoir initially contains pure water, but then the water coming in has a concentration of 10 grams/liter of a certain pollutant. The reservoir is well-stirred so that the concentration of pollutant in it is uniform at all times.

a) Set up the DE for the concentration $c = c(t)$ of salt in the reservoir at time t. Specify units.

b) Solve for $c(t)$ with the given initial condition, and graph the solution c vs. t.

c) How long will it take for the concentration of salt to be 5 g/l?

d) What happens in the long run?

Problem 6: Consider the 1st order differential equation $\frac{dy}{dx} = 2x + y$. Find a solution whose graph is also an isocline, and verify this fact analytically (i.e., by calculation, and not from a picture).

Problem 7: (a) Find the general solution of the ODE $\frac{dx}{dt} = 2t + x$ by viewing it as an inhomogeneous 1st order linear ODE. [<u>Note</u>: Except for the change in variables, this is the same problem as Problem 6.]

(b) Find the solution corresponding to the initial condition $x(0) = -2$.

Problem 8: [Direction fields, isoclines] In this problem you will study solutions of the differential equation:

$$\frac{dy}{dx} = y^2 - x .$$

Solutions of this equation do not admit expressions in terms of the standard functions of calculus, but we can study them anyway using the direction field.

(a) Draw a large pair of axes and mark off units from –4 to +4 on both. Sketch the direction field given by our equation. Do this by first sketching the isoclines for slopes $m = -1$, $m = 0$, $m = 1$, and $m = 2$. On this same graph, sketch, as best you can, a couple of solutions, using just the information given by these four isoclines.

Having done this, you will continue to investigate this equation using one of the Mathlets. So invoke http://math.mit.edu/mathlets/mathlets in a web browser and select Isoclines from the menu. (To run the applet from this window, click the little black box with a white triangle inside.) Play around with this applet for a little while. The Mathlets have many features in common, and once you get used to one it will be quicker to learn how to operate the next one. Clicking on "Help" pops up a window with a brief description of the applet's functionalities.

Select from the pull-down menu our differential equation $\frac{dy}{dx} = y^2 - x$. Move the *m* slider to $m = -2$ and release it; the $m = -2$ isocline is drawn. Do the same for $m = 0$, $m = 1$, and $m = 2$. Compare with your sketches. Then depress the mousekey over the graphing window and drag it around; you see a variety of solutions. How do they compare with what you drew earlier?

(b) A separatrix is a curve such that above it solutions behave (as *x* increases) in one way, while below it solutions behave (as *x* increases) in quite a different way. There is a separatrix for this equation such that solutions above it grow without bound (as *x* increases) while solutions below it eventually decrease (as *x* increases). Use the applet to find its graph, and submit a sketch of your result.

(c) Suppose $y(x)$ is a solution to this differential equation whose graph is tangent to the $m = -1$ isocline: it touches the $m = -1$ isocline at a point (a, b), and the two curves have the same slope at that point. Find this point on the applet, and then calculate the values of *a* and *b*.

(d) Now suppose that $y(x)$ is a solution to the equation for which $y(a) < b$, where (a, b) is the point you found in (c). What happens to it as $x \to \infty$? I claim that its graph is asymptotic to the graph of $f(x) = -\sqrt{x}$. Explain why this is so. For large *x*, is $y(x) > f(x)$, $y(x) < f(x)$, or does the answer depend on the value of $y(a)$?

The following observations will be useful in justifying your claims. Please explain as clearly as you can why each is true.

(i) The graph of $y(x)$ can't cross the $m = -1$ isocline at a point (x, y) with $x > a$.

(ii) If $c > a$ and $y(c)$ lies above the nullcline, then the graph of $y(x)$ continues to lie above the nullcline for all $x > c$.

(iii) If $c > a$ and $y(c)$ lies below the nullcline, then the graph of $y(x)$ will cross the nullcline for some $x > c$.

(e) Suppose a solution $y(x)$ has a critical point at (c, d) – that is, $y'(c) = 0$ and $y(c) = d$. What can you say about the relationship between *c* and *d*? The applet can be very helpful here, but verify your answer.

(f) It appears from the applet that all critical points are local maxima. Is that true?