9. Normalization of solutions

9.1. **Initial conditions.** The general solution of any homogeneous linear second order ODE

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0 \tag{1}$$

has the form $c_1x_1 + c_2x_2$, where c_1 and c_2 are constants. The solutions x_1, x_2 are often called "basic," but this is a poorly chosen name since it is important to understand that there is absolutely nothing special about the solutions x_1, x_2 in this formula, beyond the fact that *neither is a multiple of the other*.

For example, the ODE $\ddot{x} = 0$ has general solution $at + b$. We can take $x_1 = t$ and $x_2 = 1$ as basic solutions, and have a tendency to do this or else list them in the reverse order, so $x_1 = 1$ and $x_2 = t$. But equally well we could take a pretty randomly chosen pair of polynomials of degree at most one, such as $x_1 = 4 + t$ and $x_2 = 3 - 2t$, as basic solutions. This is because for any choice of a and b we can solve for c_1 and c_2 in $at + b = c_1x_1 + c_2x_2$. The only requirement is that neither solution is a multiple of the other. This condition is expressed by saying that the pair $\{x_1, x_2\}$ is *linearly independent*.

Given a basic pair of solutions, x_1, x_2, there is a solution of the initial value problem with $x(t_0) = a, \dot{x}(t_0) = b$, of the form $x = c_1x_1 + c_2x_2$. The constants c_1 and c_2 satisfy

$$a = x(t_0) = c_1x_1(t_0) + c_2x_2(t_0)$$

$$b = \dot{x}(t_0) = c_1\dot{x}_1(t_0) + c_2\dot{x}_2(t_0).$$

For instance, the ODE $\ddot{x} - x = 0$ has exponential solutions e^t and e^{-t}, which we can take as x_1, x_2. The initial conditions $x(0) = 2, \dot{x}(0) = 4$ then lead to the solution $x = c_1e^t + c_2e^{-t}$ as long as c_1, c_2 satisfy

$$2 = x(0) = c_1e^0 + c_2e^{-0} = c_1 + c_2,$$

$$4 = \dot{x}(0) = c_1e^0 + c_2(-e^{-0}) = c_1 - c_2,$$

This pair of linear equations has the solution $c_1 = 3, c_2 = -1$, so $x = 3e^t - e^{-t}$.

9.2. **Normalized solutions.** Very often you will have to solve the same differential equation subject to several different initial conditions. It turns out that one can solve for just *two* well chosen initial conditions, and then the solution to *any other* IVP is instantly available. Here's how.

Definition 9.2.1. A pair of solutions x_1, x_2 of (1) is *normalized at* t_0 if

$$\begin{aligned} x_1(t_0) &= 1, & x_2(t_0) &= 0, \\ \dot{x}_1(t_0) &= 0, & \dot{x}_2(t_0) &= 1. \end{aligned}$$

By existence and uniqueness of solutions with given initial conditions, there is always exactly one pair of solutions which is normalized at t_0.

For example, the solutions of $\ddot{x} = 0$ which are normalized at 0 are $x_1 = 1, x_2 = t$. To normalize at $t_0 = 1$, we must find solutions—polynomials of the form $at + b$—with the right values and derivatives at $t = 1$. These are $x_1 = 1, x_2 = t - 1$.

For another example, the "harmonic oscillator"

$$\ddot{x} + \omega_n^2 x = 0$$

has basic solutions $\cos(\omega_n t)$ and $\sin(\omega_n t)$. They are normalized at 0 only if $\omega_n = 1$, since $\frac{d}{dt}\sin(\omega_n t) = \omega_n \cos(\omega_n t)$ has value ω_n at $t = 0$, rather than value 1. We can fix this (as long as $\omega_n \neq 0$) by dividing by ω_n: so

$$\cos(\omega_n t)\,, \qquad \omega_n^{-1}\sin(\omega_n t) \tag{2}$$

is the pair of solutions to $\ddot{x} + \omega_n^2 x = 0$ which is normalized at $t_0 = 0$.

Here is another example. The equation $\ddot{x} - x = 0$ has linearly independent solutions e^t, e^{-t}, but these are not normalized at any t_0 (for example because neither is ever zero). To find x_1 in a pair of solutions normalized at $t_0 = 0$, we take $x_1 = ae^t + be^{-t}$ and find a, b such that $x_1(0) = 1$ and $\dot{x}_1(0) = 0$. Since $\dot{x}_1 = ae^t - be^{-t}$, this leads to the pair of equations $a + b = 1, a - b = 0$, with solution $a = b = 1/2$. To find $x_2 = ae^t + be^{-t}$ $x_2(0) = 0, \dot{x}_2(0) = 1$ imply $a + b = 0, a - b = 1$ or $a = 1/2, b = -1/2$. Thus our normalized solutions x_1 and x_2 are the *hyperbolic sine* and *cosine* functions:

$$\cosh t = \frac{e^t + e^{-t}}{2}, \qquad \sinh t = \frac{e^t - e^{-t}}{2}\,.$$

These functions are important precisely because they occur as normalized solutions of $\ddot{x} - x = 0$.

Normalized solutions are always linearly independent: x_1 can't be a multiple of x_2 because $x_1(t_0) \neq 0$ while $x_2(t_0) = 0$, and x_2 can't be a multiple of x_1 because $\dot{x}_2(t_0) \neq 0$ while $\dot{x}_1(t_0) = 0$.

Now suppose we wish to solve (1) with the general initial conditions.

If x_1 and x_2 are a pair of solutions normalized at t_0, then the solution x with $x(t_0) = a$, $\dot{x}(t_0) = b$ is $$x = ax_1 + bx_2 \,.$$

The constants of integration *are* the initial conditions.

If I want x such that $\ddot{x} + x = 0$ and $x(0) = 3, \dot{x}(0) = 2$, for example, we have $x = 3\cos t + 2\sin t$. Or, for an other example, the solution of $\ddot{x} - x = 0$ for which $x(0) = 2$ and $\dot{x}(0) = 4$ is $x = 2\cosh(t) + 4\sinh(t)$. You can check that this is the same as the solution given above.

Exercise 9.2.2. Check the identity

$$\cosh^2 t - \sinh^2 t = 1\,.$$

9.3. **ZSR and ZIR.** There is an interesting way to decompose the solution of a linear initial value problem which is appropriate to the *inhomogeneous* case and which arises in the system/signal approach. Two distinguishable bits of data determine the choice of solution: the initial condition, and the input signal.

Suppose we are studying the initial value problem

$$\ddot{x} + p(t)\dot{x} + q(t)x = f(t)\,, \quad x(t_0) = x_0\,, \quad \dot{x}(t_0) = \dot{x}_0\,. \tag{3}$$

There are two related initial value problems to consider:

[**ZSR**] The *same* ODE but with *rest* initial conditions (or "zero state"):

$$\ddot{x} + p(t)\dot{x} + q(t)x = f(t)\,, \quad x(t_0) = 0\,, \quad \dot{x}(t_0) = 0\,.$$

Its solution is called the **Zero State Response** or **ZSR.** It depends entirely on the input signal, and assumes zero initial conditions. We'll write x_f for it, using the notation for the input signal as subscript.

[**ZIR**] The associated *homogeneous* ODE with the *given* initial conditions:

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0\,, \quad x(t_0) = x_0\,, \quad \dot{x}(t_0) = \dot{x}_0\,.$$

Its solution is called the the **Zero Input Response**, or **ZIR**. It depends entirely on the initial conditions, and assumes null input signal. We'll write x_h for it, where h indicates "homogeneous."

By the superposition principle, the solution to (3) is precisely

$$x = x_f + x_h.$$

The solution to the initial value problem (3) is the sum of a ZSR and a ZIR, in exactly one way.

Example 9.3.1. Drive a harmonic oscillator with a sinusoidal signal:

$$\ddot{x} + \omega_n^2 x = a\cos(\omega t)$$

(so $f(t) = a\cos(\omega t)$) and specify initial conditions $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$. Assume that the system is not in resonance with the signal, so $\omega \neq \omega_n$. Then the Exponential Response Formula (Section 10) shows that the general solution is

$$x = a\frac{\cos(\omega t)}{\omega_n^2 - \omega^2} + b\cos(\omega_n t) + c\sin(\omega_n t)$$

where b and c are constants of integration. To find the ZSR we need to find $\dot{x}$, and then arrange the constants of integration so that both $x(0) = 0$ and $\dot{x}(0) = 0$. Differentiate to see

$$\dot{x} = -a\omega\frac{\sin(\omega t)}{\omega_n^2 - \omega^2} - b\omega_n\sin(\omega_n t) + c\omega_n\cos(\omega_n t)$$

so $\dot{x}(0) = c\omega_n$, which can be made zero by setting $c = 0$. Then $x(0) = a/(\omega_n^2 - \omega^2) + b$, so $b = -a/(\omega_n^2 - \omega^2)$, and the ZSR is

$$x_f = a\,\frac{\cos(\omega t) - \cos(\omega_n t)}{\omega_n^2 - \omega^2}\,.$$

The ZIR is

$$x_h = b\cos(\omega_n t) + c\sin(\omega_n t)$$

where this time b and c are chosen so that $x_h(0) = x_0$ and $\dot{x}_h(0) = \dot{x}_0$. Thus (using (2) above)

$$x_h = x_0\cos(\omega_n t) + \dot{x}_0\frac{\sin(\omega_n t)}{\omega_n}\,.$$

Example 9.3.2. The same works for linear equations of any order. For example, the solution to the bank account equation (Section 2)

$$\dot{x} - Ix = c\,, \quad x(0) = x_0,$$

(where we'll take the interest rate I and the rate of deposit c to be constant, and $t_0 = 0$) can be written as

$$x = \frac{c}{I}(e^{It} - 1) + x_0 e^{It}.$$

The first term is the ZSR, depending on c and taking the value 0 at $t = 0$. The second term is the ZIR, a solution to the homogeneous equation depending solely on x_0.