8. RLC circuits

8.1. **Series RLC Circuits.** Electric circuits provide an important example of linear, time-invariant differential equations, alongside mechanical systems. We will consider only the simple series circuit pictured below.

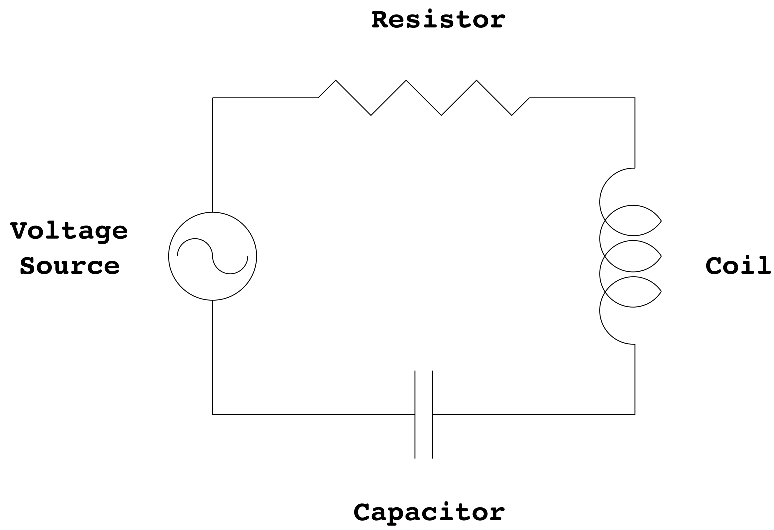

Figure 5. Series RLC Circuit

The Mathlet `Series RLC Circuit` exhibits the behavior of this system, when the voltage source provides a sinusoidal signal.

Current flows through the circuit; in this simple loop circuit the current through any two points is the same at any given moment. Current is denoted by the letter I, or $I(t)$ since it is generally a function of time.

The current is created by a force, the "electromotive force," which is determined by voltage *differences*. The voltage *drop* across a component of the system *except for the power source* will be denoted by V with a subscript. Each is a function of time. If we orient the circuit consistently, say clockwise, then we let

$V_L(t)$ denote the voltage drop across the coil
$V_R(t)$ denote the voltage drop across the resistor
$V_C(t)$ denote the voltage drop across the capacitor
$V(t)$ denote the voltage *increase* across the power source

"Kirchoff's Voltage Law" then states that

$$V = V_L + V_R + V_C \tag{1}$$

The circuit components are characterized by the relationship between the current flowing through them and the voltage drop across them:

$$\begin{aligned} \text{Coil}: &\quad V_L = L\dot{I} \\ \text{Resistor}: &\quad V_R = RI \\ \text{Capacitor}: &\quad \dot{V}_C = (1/C)I \end{aligned} \tag{2}$$

The constants here are the "inductance" L of the coil, the "resistance" R of the resistor, and the *inverse* of the "capacitance" C of the capacitor. A very large capacitor, with C large, is almost like no capacitor at all; electrons build up on one plate, and push out electrons on the other, to form an uninterrupted circuit. We'll say a word about the actual units below.

To get the expressions (2) into comparable form, differentiate the first two. Differentiating (1) gives $\dot{V}_L + \dot{V}_R + \dot{V}_C = \dot{V}$, and substituting the values for $\dot{V}_L$, $\dot{V}_R$, and $\dot{V}_C$ gives us

$$\boxed{L\ddot{I} + R\dot{I} + (1/C)I = \dot{V}} \tag{3}$$

This equation describes how I is determined from the impressed voltage V. It is a second order linear time invariant ODE. Comparing it with the familiar equation

$$\boxed{m\ddot{x} + b\dot{x} + kx = F} \tag{4}$$

governing the displacement in a spring-mass-dashpot system reveals an analogy between the two types of system:

Mechanical	**Electrical**
Mass	Coil
Damper	Resistor
Spring	Capacitor
Driving force	Time derivative of impressed voltage
Displacement	Current

8.2. **A word about units.** There is a standard system of units called the International System of Units, SI, formerly known as the mks (meter-kilogram-second) system. In terms of those units, (3) is correct when:

inductance L is measured in henries, H
resistance R is measured in ohms, Ω
capacitance C is measured in farads, F
voltgage V is measured in volts, also denoted V
current I is measured in amperes, A

Balancing units in the equation shows that

$$\frac{\text{henry} \cdot \text{ampere}}{\text{sec}^2} = \frac{\text{ohm} \cdot \text{ampere}}{\text{sec}} = \frac{\text{ampere}}{\text{farad}} = \frac{\text{volt}}{\text{sec}}$$

Thus one henry is the same as one volt-second per ampere.

The analogue for mechanical units is this:

mass m is measured in kilograms, kg
damping constant b is measured in kg/sec
spring constant k is measured in kg/sec^2
applied force F is measured in newtons, N
displacement x is measured in meters, m

Here

$$\text{newton} = \frac{\text{kg} \cdot \text{m}}{\text{sec}^2}$$

so another way to describe the units in which the spring constant is measured in is as newtons per meter—the amount of force it produces when stretched by one meter.