

## 5. THE ALGEBRA OF COMPLEX NUMBERS

We use complex numbers for more purposes in this course than the textbook does. This chapter tries to fill some gaps.

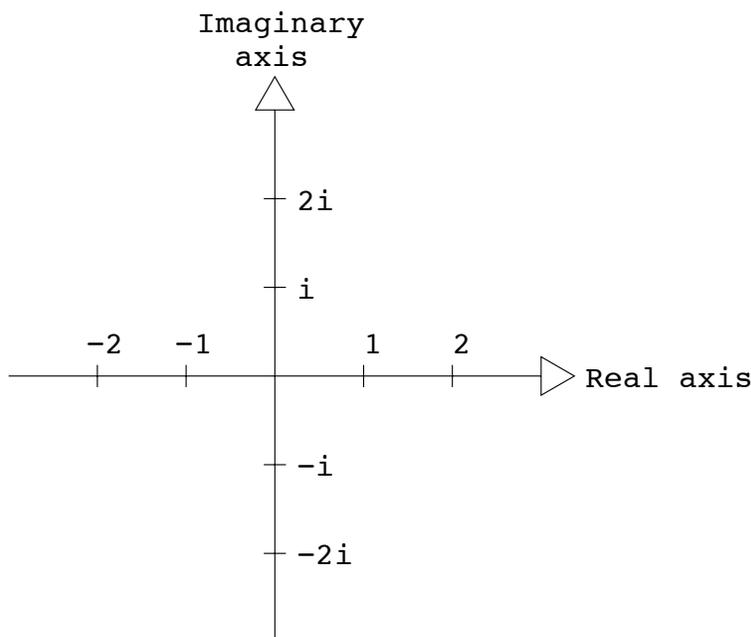
**5.1. Complex algebra.** A “complex number” is an element  $(a, b)$  of the plane.

Special notation is used for vectors in the plane when they are thought of as complex numbers. We think of the real numbers as lying in the plane as the horizontal axis: the real number  $a$  is identified with the vector  $(a, 0)$ . In particular 1 is the basis vector  $(1, 0)$ .

The vector  $(0, 1)$  is given the symbol  $i$ . Every element of the plane is a linear combination of these two vectors, 1 and  $i$ :

$$(a, b) = a + bi.$$

When we think of a point in the plane as a complex number, we always write  $a + bi$  rather than  $(a, b)$ .



The real number  $a$  is called the **real part** of  $a + bi$ , and the real number  $b$  is the **imaginary part** of  $a + bi$ . Notation:

$$\operatorname{Re}(a + bi) = a, \quad \operatorname{Im}(a + bi) = b.$$

A complex number is **purely imaginary** if it lies on the vertical or **imaginary axis**. It is a real multiple of the complex number  $i$ . A

complex number is **real** if it lies on the horizontal or **real axis**. It is a real multiple of the complex number 1.

The only complex number which is both real and purely imaginary is 0, the origin.

Complex numbers are **added** by vector addition. Complex numbers are **multiplied** by the rule

$$i^2 = -1$$

and the standard rules of arithmetic.

This means we “FOIL” out products. For example,

$$(1 + 2i)(3 + 4i) = 1 \cdot 3 + 1 \cdot 4i + (2i) \cdot 3 + (2i) \cdot (4i) = \dots$$

—and then use commutativity and the rule  $i^2 = -1$ —

$$\dots = 3 + 4i + 6i - 8 = -5 + 10i.$$

The real part of the product is the product of real parts *minus* the product of imaginary parts. The imaginary part of the product is the sum of the crossterms.

We will write the set of all real numbers as  $\mathbb{R}$  and the set of all complex numbers as  $\mathbb{C}$ . Often the letters  $z$ ,  $w$ ,  $v$ , and  $s$ , and  $r$  are used to denote complex numbers. The operations on complex numbers satisfy the usual rules:

**Theorem.** If  $v$ ,  $w$ , and  $z$  are complex numbers then

$$z + 0 = z, \quad v + (w + z) = (v + w) + z, \quad w + z = z + w,$$

$$z \cdot 1 = z, \quad v(wz) = (vw)z, \quad wz = zw$$

$$(v + w)z = vz + wz.$$

This is easy to check. The vector negative gives an additive inverse, and, as we will see below, every complex number except 0 has a multiplicative inverse.

Unlike the real numbers, the set of complex numbers doesn’t come with a notion of greater than or less than.

**Exercise 5.1.1.** Rewrite  $((1 + \sqrt{3}i)/2)^3$  and  $(1 + i)^4$  in the form  $a + bi$ .

**5.2. Conjugation and modulus.** The **complex conjugate** of a complex number  $a + bi$  is the complex number  $a - bi$ . Geometrically, complex conjugation reflects a complex number across the real axis. The complex conjugate of  $z$  is written  $\bar{z}$ :

$$\boxed{a + bi = a - bi}$$

**Theorem.** Complex conjugation satisfies:

$$\bar{\bar{z}} = z, \quad \overline{w + z} = \bar{w} + \bar{z}, \quad \overline{wz} = \bar{w}\bar{z}.$$

A complex number  $z$  is real exactly when  $\bar{z} = z$  and is purely imaginary exactly when  $\bar{z} = -z$ . The real and imaginary parts of a complex number  $z$  can be written using complex conjugation as

$$(1) \quad \operatorname{Re} z = \frac{z + \bar{z}}{2}, \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i}.$$

Again this is easy to check.

**Exercise 5.2.1.** Show that if  $z = a + bi$  then

$$z\bar{z} = a^2 + b^2.$$

This is the square of the distance from the origin, and so is a nonnegative real number, nonzero as long as  $z \neq 0$ . Its nonnegative square root is the **absolute value** or **modulus** or **magnitude** of  $z$ , written

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}.$$

Thus

$$(2) \quad \boxed{z\bar{z} = |z|^2}$$

**Exercise 5.2.2.** Show that  $|wz| = |w||z|$ . Since this notation clearly extends its meaning on real numbers, it follows that if  $r$  is a positive real number then  $|rz| = r|z|$ , in keeping with the interpretation of absolute value as distance from the origin.

Any nonzero complex number has a multiplicative inverse: as  $z\bar{z} = |z|^2$ ,  $z^{-1} = \bar{z}/|z|^2$ . If  $z = a + bi$ , this says

$$\boxed{\frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2}}$$

This is “rationalizing the denominator.”

**Exercise 5.2.3.** Compute  $i^{-1}$ ,  $(1 + i)^{-1}$ , and  $\frac{1 + i}{2 - i}$ . What is  $|z^{-1}|$  in terms of  $|z|$ ?

**Exercise 5.2.4.** Since rules of algebra hold for complex numbers as well as for real numbers, the quadratic formula correctly gives the roots of a quadratic equation  $x^2 + bx + c = 0$  even when the “discriminant”  $b^2 - 4c$  is negative. What are the roots of  $x^2 + x + 1$ ? Of  $x^2 + x + 2$ ? The quadratic formula even works if  $b$  and  $c$  are not real. Solve  $x^2 + ix + 1 = 0$ .

The modulus or magnitude of  $|z|$  of a complex number is part of the polar coordinate description of the point  $z$ . The other polar coordinate—the polar angle—has a name too; it is the **argument** or **angle** of  $z$ , and is written  $\text{Arg}(z)$ . Trigonometry shows that

$$(3) \quad \tan(\text{Arg}(z)) = \frac{\text{Im } z}{\text{Re } z}$$

As usual, the argument is well defined only up to adding multiples of  $2\pi$ , and it’s not defined at all for the complex number 0.

**5.3. The fundamental theorem of algebra.** Complex numbers remedy a defect of real numbers, by providing a solution for the quadratic equation  $x^2 + 1 = 0$ . It turns out that you don’t have to worry that someday you’ll come across a weird equation that requires numbers even more complex than complex numbers:

**Fundamental Theorem of Algebra.** Any nonconstant polynomial (even one with complex coefficients) has a complex root.

Once you have a single root, say  $r$ , for a polynomial  $p(x)$ , you can divide through by  $(x - r)$  and get a polynomial of smaller degree as quotient, which then also has a complex root, and so on. The result is that a polynomial  $p(x) = ax^n + \cdots$  of degree  $n$  (so  $a \neq 0$ ) factors completely into linear factors over the complex numbers:

$$p(x) = a(x - r_1)(x - r_2) \cdots (x - r_n).$$