4. Sinusoidal solutions

Many things in nature are periodic, even sinusoidal. We will begin by reviewing terms surrounding periodic functions. If an LTI system is fed a periodic input signal, we have a right to hope for a periodic solution. Usually there is exactly one periodic solution, and often all other solutions differ from it by a "transient," a function that dies off exponentially. This section begins by setting out terms and facts about periodic and sinusoidal functions, and then studies the response of a first order LTI system to a sinusoidal signal. This is a special case of a general theory described in Sections 10 and 15.

4.1. **Periodic and sinusoidal functions.** A function $f(t)$ is **periodic** if there is a number $a > 0$ such that

$$f(t+a) = f(t)$$

for all t. It repeats itself over and over, and has done since the world began. The number a is a **period**. Notice that if a is a period then so is $2a$, and $3a$, and so in fact is any positive integral multiple of a. If $f(t)$ is continuous and not constant, there is a smallest period, called the *minimal period* or simply *the* period, and is often denoted by P. If the independent variable t is a distance rather than a time, the period is also called the *wavelength*, and denoted in physics by the Greek letter "lambda," λ.

A periodic function of time has a *frequency*, too, often denoted by f or by the Greek letter "nu," ν. The frequency is the reciprocal of the minimal period:

$$\nu = 1/P.$$

This is the number of cycles per unit time, and its units are, for example, $(\mathrm{sec})^{-1}$.

Since many periodic functions are closely related to sine and cosines, it is common to use the **angular frequency** denoted by the Greek letter "omega," ω. This is 2π times the frequency:

$$\omega = 2\pi\nu.$$

If ν is the number of *cycles per second*, then ω is the number of *radians per second*. In terms of the angular frequency, the period is

$$P = \frac{2\pi}{\omega}.$$

The **sinusoidal functions** make up a particular class of periodic functions, namely, those which can be expressed as a cosine function which as been *amplified, shifted* and *compressed*:

$$\boxed{f(t) = A\cos(\omega t - \phi)} \tag{1}$$

The function (1) is periodic of period $2\pi/\omega$, frequency $\omega/2\pi$, and angular frequency ω.

The parameter A (or, better, $|A|$) is the **amplitude** of (1). By replacing ϕ by $\phi + \pi$ if necessary, we may always assume $A \geq 0$, and we will usually make this assumption.

The number ϕ is the **phase lag** (relative to the cosine). It is measured in radians or degrees. The **phase shift** is $-\phi$. In many applications, $f(t)$ represents the response of a system to a signal of the form $B\cos(\omega t)$. The phase lag is then usually positive—the system response lags behind the signal—and this is one reason why we choose to favor the *lag* and not the *shift* by assigning a notation to it. Some engineers prefer to use ϕ for the phase shift, i.e. the negative of our ϕ. You will just have to check and see which convention is in use.

The phase lag can be chosen to lie between 0 and 2π. The ratio $\phi/2\pi$ is the fraction of a full period by which the function (1) is shifted to the right relative to $\cos(\omega t)$: $f(t)$ is $\phi/2\pi$ radians behind $\cos(\omega t)$.

Here are the instructions for building the graph of (1) from the graph of $\cos t$. First *amplify*, or vertically expand, the graph by a factor of A; then *shift* the result to the right by ϕ units; and finally *compress* it horizontally by a factor of ω.

One can also write (1) as

$$f(t) = A\cos(\omega(t - t_0)),$$

where $\omega t_0 = \phi$, or

$$t_0 = \frac{\phi}{2\pi} P \tag{2}$$

t_0 is the **time lag**. It is measured in the same units as t, and represents the amount of time $f(t)$ lags behind the compressed cosine signal $\cos(\omega t)$. Equation (2) expresses the fact that t_0 makes up the same fraction of the period P as the phase lag ϕ does of the period of the cosine function.

Sinusoidal functions observe an amazing closure property: *Any linear combination of sinusoids with the same frequency is another sinusoid with that frequency.*

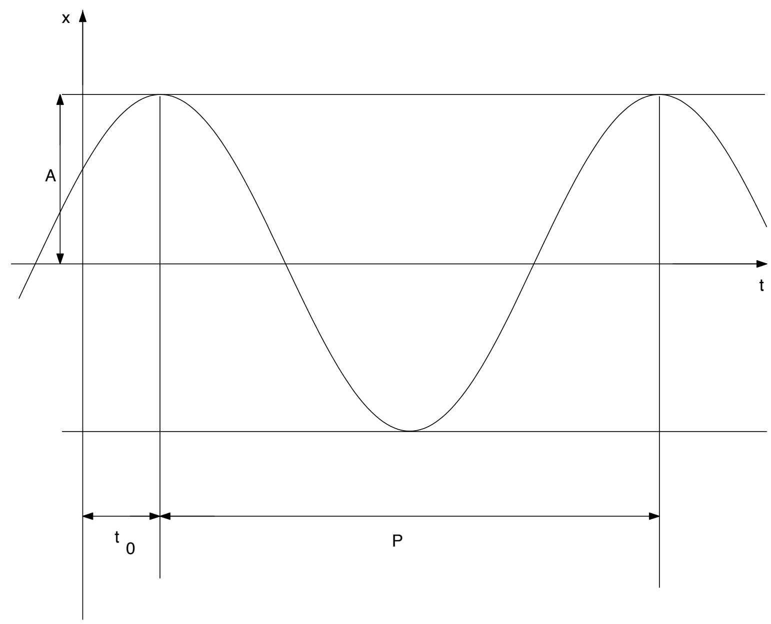


FIGURE 1. Parameters of a sinusoidal function

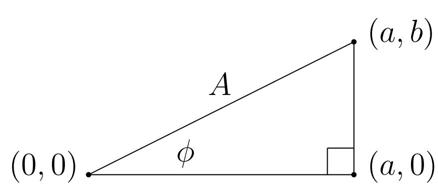
As one case, the linear combination $a\cos(\omega t) + b\sin(\omega t)$ can be written as $A\cos(\omega t - \phi)$ for some amplitude A and phase lag ϕ:

$$A\cos(\omega t - \phi) = a\cos(\omega t) + b\sin(\omega t) \tag{3}$$

where A and ϕ are the polar coordinates of the point with rectangular coordinates (a, b); that is,

$$a = A\cos(\phi)\,, \quad b = A\sin(\phi)$$

This is the familiar formula for the cosine of a difference. Geometrically:



In (3) either or both of a and b can be negative; (a, b) can be any point in the plane. This identity is well illustrated by the Mathlet `Trigonometric Identity`.

I want to stress the importance of this simple observation. Remember:

> In (3), A and ϕ are the polar coordinates of (a, b)

If we replace ωt by $-\omega t+\phi$ in (3), then $\omega t-\phi$ gets replaced by $-\omega t$ and the identity becomes $A\cos(-\omega t) = a\cos(-\omega t+\phi)+b\sin(-\omega t+\phi)$. Since the cosine is even and the sine is odd, this is equivalent to

$$A\cos(\omega t) = a\cos(\omega t-\phi) - b\sin(\omega t-\phi) \tag{4}$$

which is sometimes useful as well. The relationship between a, b, A, and ϕ is always the same.

4.2. **Periodic solutions and transients.** Let's return to the model of the cooler, described in Section 2.2: $x(t)$ is the temperature inside the cooler, $y(t)$ the temperature outside, and we model the cooler by the first order linear equation with constant coefficient:

$$\dot{x} + kx = ky.$$

Let's suppose the outside temperature varies sinusoidally (warmer in the day, cooler at night). (This involves choosing units for temperature so that the *average* temperature is zero.) By setting our clock so that the highest temperature occurs at $t = 0$, we can thus model $y(t)$ by

$$y(t) = y_0\cos(\omega t)$$

where $y_0 = y(0)$ is the daily high temperature. So our model is

$$\dot{x} + kx = ky_0\cos(\omega t). \tag{5}$$

The equation (5) can be solved by the standard method for solving first order linear ODEs (integrating factors, or variation of parameter). In fact, we will see in Section 10 that since the right hand side is sinusoidal there is an explicit and direct way to write down the solution using complex numbers. Here's a different approach, which one might call the "method of optimism."

Let's look for a *periodic* solution; not unreasonable since the driving function is periodic. Even more optimistically, let's hope for a sinusoidal function. At first you might hope that $A\cos(\omega t)$ would work, for suitable constant A, but that turns out to be too much to ask, and doesn't reflect what we already know from our experience with temperature: The temperature inside the cooler tends to lag behind the ambient temperature. This lag can be accommodated by means of the formula:

$$x_p = gy_0\cos(\omega t-\phi). \tag{6}$$

We have chosen to write the amplitude here as a multiple of the ambient high temperature y_0. The multiplier g and the phase lag ϕ are numbers which we will try to choose so that x_p is indeed a solution. We use the subscript p to indicate that this is a Particular solution. It is also a Periodic solution, and generally will turn out to be the only periodic solution.

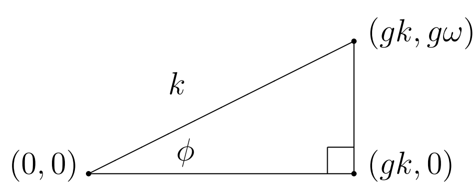
We can and will take ϕ between 0 and 2π, and $g \geq 0$; so gy_0 is the amplitude of the temperature oscillation in the cooler. The number g is the ratio of the maximum temperature in the cooler to the maximum ambient temperature; it is called the **gain** of the system. The angle ϕ is the **phase lag**. Both of these quantities depend upon the coupling constant k and the angular frequency of the input signal ω.

To see what g and ϕ must be in order for x_p to be a solution, we will use the alternate form (4) of the trigonometric identity. The important thing here is that there is only one pair of numbers (a, b) for which this identity holds: They are the rectangular coordinates of the point with polar coordinates (A, ϕ).

If $x = gy_0 \cos(\omega t - \phi)$, then $\dot{x} = -gy_0\omega \sin(\omega t - \phi)$. Substitute these values into the ODE:

$$gy_0 k \cos(\omega t - \phi) - gy_0\omega \sin(\omega t - \phi) = ky_0 \cos(\omega t).$$

I have switched the order of the terms on the left hand side, to make comparison with the trig identity (4) easier. Cancel the y_0. Comparing this with (4), we get the triangle



From this we read off

$$\tan \phi = \omega / k \tag{7}$$

and

$$g = \frac{k}{\sqrt{k^2 + \omega^2}} = \frac{1}{\sqrt{1 + (\omega/k)^2}}. \tag{8}$$

Our work shows that with these values for g and ϕ the function x_p given by (6) *is* a solution to (5).

Incidentally, the triangle shows that the gain g and the phase lag ϕ in this first order equation are related by

$$g = \cos \phi. \tag{9}$$

According to the principle of superposition, the general solution is

$$x = x_p + ce^{-kt}, \tag{10}$$

since e^{-kt} is a nonzero solution of the homogeneous equation $\dot{x} + kx = 0$.

You can see why you need the extra term ce^{-kt}. Putting $t = 0$ in (6) gives a specific value for $x(0)$. We have to do something to build a solution for initial value problems specifying different values for $x(0)$, and this is what the additional term ce^{-kt} is for. But this term dies off exponentially with time, and leaves us, for large t, with the same solution, x_p, independent of the initial conditions. In terms of the model, the cooler did start out at refrigerator temperature, far from the "steady state." In fact the periodic system response has average value zero, equal to the average value of the signal. No matter what the initial temperature $x(0)$ in the cooler, as time goes by the temperature function will converge to $x_p(t)$. This long-term lack of dependence on initial conditions confirms an intuition. The exponential term ce^{-kt} is called a **transient**. The general solution, in this case and in many others, is a periodic solution plus a transient.

I stress that *any* solution can serve as a "particular solution." The solution x_p we came up with here is special not because it's a particular solution, but rather because it's a *periodic solution*. In fact (assuming $k > 0$) it's the *only* periodic solution.

4.3. **Amplitude and phase response.** There is a lot more to learn from the formula (6) and the values for g and ϕ given in (7) and (8). The terminology applied below to solutions of the first order equation (5) applies equally well to solutions of second and higher order equations. See Section 15 for further discussion, and the Mathlet `Amplitude and Phase: First Order` for a dynamic illustration.

Let's fix the coupling constant k and think about how g and ϕ vary as we vary ω, the angular frequency of the signal. Thus we will regard them as functions of ω, and we may write $g(\omega)$ and $\phi(\omega)$ in order to emphasize this perspective. We are supposing that the *system* is constant, and watching its response to a variety of different input signals. Graphs of $g(\omega)$ and $-\phi(\omega)$ for values of the coupling constant $k = .25, .5, .75, 1, 1.25, 1.5$ is displayed in Figure 2.

These graphs are essentially **Bode plots**. Technically, the Bode plots displays $\log g(\omega)$ and $-\phi(\omega)$ against $\log \omega$.

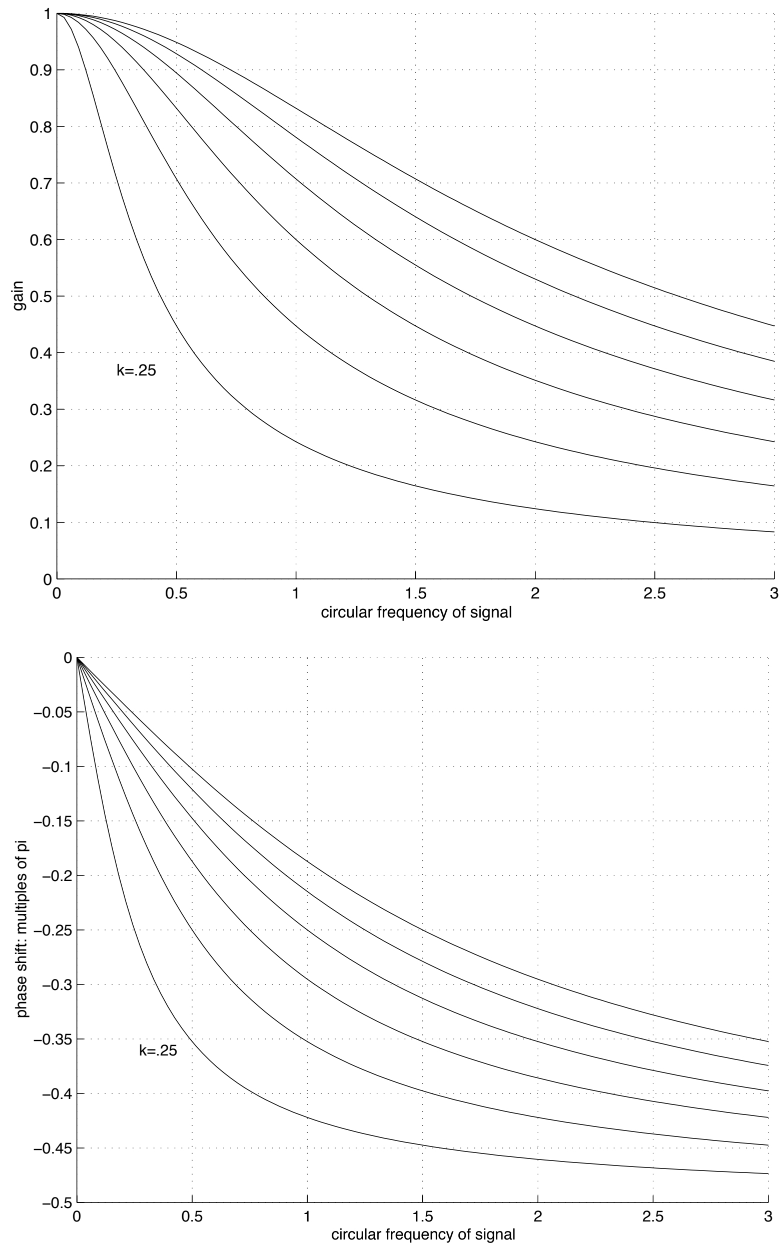


FIGURE 2. First order frequency response curves