29. The Laplace transform and more general systems

This section gives a hint of how flexible a device the Laplace transform is in engineering applications.

29.1. **Zeros of the Laplace transform: stillness in motion.** The mathematical theory of functions of a complex variable shows that the *zeros* of $F(s)$—the values r of s for which $F(r) = 0$—are just as important to our understanding of it as are the poles. This symmetry is reflected in engineering as well; the location of the zeros of the transfer function has just as much significance as the location of the poles. Instead of recording resonance, they reflect stillness.

We envision the following double spring system: there is an object with mass m_1 suspended by a spring with spring constant k_1. A second object with mass m_2 is suspended from this first object by a second spring with constant k_2. The system is driven by motion of the top of the top spring according to a function $f(t)$. Pick coordinates so that x_1 is the position of the first object and x_2 is the position of the second, both increasing in the downward direction, and such that when $f(t) = x_1 = x_2 = 0$ the springs exert no force.

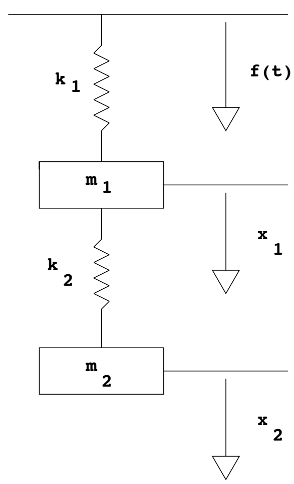

Figure 17. Two spring system

The equations of motion are

$$\text{(1)} \qquad \begin{cases} m_1\ddot{x}_1 &=& k_1(f(t) - x_1) - k_2(x_1 - x_2) \\ m_2\ddot{x}_2 &=& k_2(x_1 - x_2) \end{cases}$$

This is a *system of second order equations*, and as you can imagine mechanical engineering is full of similar systems.

Suppose that our main interest is in x_1. Let's take Laplace transforms, and assume rest initial conditions.

$$\begin{cases} (m_1 s^2 + (k_1 + k_2))X_1 &= k_2 X_2 + k_1 F \\ (m_2 s^2 + k_2)X_2 &= k_2 X_1. \end{cases}$$

Use the second equation to express X_2 in terms of X_1, and substitute this value into the first equation. Then solve for X_1 to get:

$$X_1(s) = \frac{m_2 s^2 + k_2}{(m_1 s^2 + (k_1 + k_2))(m_2 s^2 + k_2) - k_2^2} \cdot k_1 F(s).$$

The "transfer function" $W(s)$ is then the ratio of the LT of the system response, X_1, and the LT of the input signal, F:

$$W(s) = \frac{k_1(m_2 s^2 + k_2)}{(m_1 s^2 + (k_1 + k_2))(m_2 s^2 + k_2) - k_2^2}.$$

It is still the case that $W(r)$ is the multiple of e^{rt} which occurs as x_1 in a solution to the equations (1) when we take $f(t) = e^{rt}$. Thus the zeros of $W(s)$ at $s = \pm i\sqrt{k_2/m_2}$—the values of s for which $W(s) = 0$—reflect a "neutralizing" angular frequency of $\omega = \sqrt{k_2/m_2}$. If $f(t)$ is sinusoidal of this angular frequency then $x_1 = 0$ is a solution. The suspended weight oscillates with (k_1/k_2) times the amplitude of $f(t)$ and reversed in phase (independent of the masses!), and exactly cancels the impressed force. Check it out!

29.2. **General LTI systems.** The weight function $w(t)$, or its Laplace transform, the transfer function $W(s)$, completely determine the system. The transfer function of an ODE has a very restricted form—it is the reciprocal of a polynomial; but the mechanism for determining the system response makes sense for much more general complex functions $W(t)$, and, correspondingly, much more general "weight functions" $w(t)$: given a very general function $w(t)$, we can define an LTI system by declaring that a signal $f(t)$ results in a system response (with null initial condition, though in fact nontrivial initial conditions can be handled too, by absorbing them into the signal using delta functions) given by the convolution $f(t) * w(t)$. The apparatus of the Laplace transform helps us, too, since we can compute this system response as the inverse Laplace transform of $F(s)W(s)$. This mechanism allows us to represent the *system*, the *signal*, and the *system response*, all three, using *functions* (of t, or of s). Differential operators have vanished from the scene. This flexibility results in a tool of tremendous power.