# 27. The pole diagram and the Laplace transform

When working with the Laplace transform, it is best to think of the variable $s$ in $F(s)$ as ranging over the *complex* numbers. In the first section below we will discuss a way of visualizing at least some aspects of such a function—via the "pole diagram." Next we'll describe what the pole diagram of $F(s)$ tells us—and what it does not tell us—about the original function $f(t)$. In the third section we discuss the properties of the integral defining the Laplace transform, allowing $s$ to be complex. The last section describes the Laplace transform of a periodic function of $t$, and its pole diagram, linking the Laplace transform to Fourier series.

27.1. **Poles and the pole diagram.** The real power of the Laplace transform is not so much as an algorithm for explicitly computing linear time-invariant system responses as in gaining insight into these responses *without* explicitly computing them. (A further feature of the Laplace transform is that it allows one to analyze systems which are not modeled by ODEs at all, by exactly the same methodology.) To achieve this insight we will have to regard the transform variable $s$ as *complex*, and the transform function $F(s)$ as a complex-valued function of a complex variable.

A simple example is $F(s) = 1/(s-z)$, for a fixed complex number $z$. We can get some insight into a complex-valued function of a complex variable, such as $1/(s-z)$, by thinking about its absolute value: $|1/(s-z)| = 1/|s-z|$. This is now a *real-valued* function on the complex plane, and its graph is a surface lying over the plane, whose height over a point $s$ is given by the value $|1/(s-z)|$. This is a tent-like surface lying over the complex plane, with elevation given by the reciprocal of the distance to $z$. It sweeps up to infinity like a hyperbola as $s$ approaches $z$; it's as if it is being held up at $s = z$ by a tent-pole, and perhaps this is why we say that $1/(s-z)$ "has a pole at $s = z$." Generally, a function of complex numbers has a "pole" at $s = z$ when it becomes infinite there.

$F(s) = 1/(s-z)$ is an example of a **rational function**: a quotient of one polynomial by another. The Laplace transforms of many important functions are rational functions, and we will start by discussing rational functions.

A product of two rational functions is again a rational function. Because you can use a common denominator, a sum of two rational functions is also a rational function. The reciprocal of any rational function except the zero function is again a rational function—exchange numerator and denominator. In these algebraic respects, the collection of rational functions behaves like the set of rational *numbers.* Also like rational numbers, you can simplify the fraction by cancelling terms in numerator and denominator, till the two don't have any common factors. (In the case of rational numbers, you do have to allow $\pm 1$ as a common factor! In the case of rational functions, you do have to allow nonzero contants as common factors.)

When written in reduced form, the magnitude of $F(s)$ blows up to $\infty$ as $s$ approaches a root of the denominator. The complex roots of the denominator are the **poles** of $F(s)$.

In case the denominator doesn't have any repeated roots, partial fractions let you write $F(s)$ as

$$F(s) = p(s) + \frac{w_1}{s - z_1} + \cdots + \frac{w_n}{s - z_n} \tag{1}$$

where $p(s)$ is a polynomial, $z_1, \ldots z_n$ are complex constants, and $w_1, \ldots, w_n$ are nonzero complex constants..

For example, the calculation done in Section 25.5 shows that the poles of $F(s) = 1/(s^3 + s^2 - 2)$ are at $s = 1$, $s = -1 + i$, and $s = -1 - i$.

The **pole diagram** of a complex function $F(s)$ is just the complex plane with the poles of $F(s)$ marked on it. Figure 15 shows the pole diagram of the function $F(s) = 1/(s^3 + s^2 - 2)$.

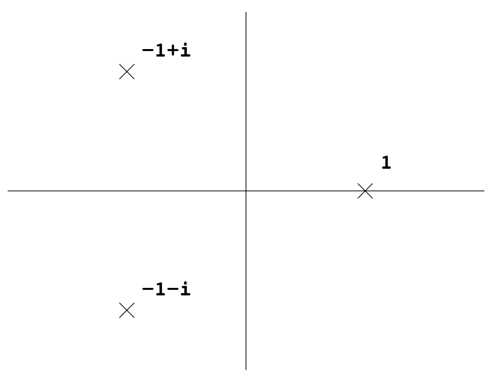


FIGURE 15. Pole diagram for $1/(s^3 + s^2 - 2)$

The constant $w_k$ appearing in (1) is the **residue** of the pole at $s = z_k$. The calculation in 25.5 shows that the residue at $s = 1$ is $1/5$, the residue at $s = -1 + 2i$ is $(-1 + 2i)/10$, and the residue at $s = -1 - 2i$ is $(-1 - 2i)/10$. These are the numerators in (25.3). They are also the coefficients that appear in front of the exponential functions of $t$ appearing in the inverse Laplace transform of $F(s)$.

Laplace transforms are not always rational functions. For example, the exponential function occurs: $F(s) = e^{-as}$ is the Laplace transform of $\delta(t-a)$, for example. Generally, a pole of a function of the complex number $s$ is a value of $s$ near which the function blows up to infinite value. The exponential function has *no poles*: it takes on well defined complex values for any complex input $s$.

We can form more elaborate complex functions by taking products—$e^{-s}/(s^3+s^2-2)$, for example. The numerator doesn't contribute any poles. Nor does it kill any poles—it is never zero, so it doesn't cancel any of the roots of the denominator. The pole diagram of this function is the same as the pole diagram of $1/(s^3+s^2-2)$.

A complex function is by no means completely specified by its pole diagram. Nevertheless, the pole diagram of $F(s)$ carries a lot of information about $F(s)$, and if $F(s)$ is the Laplace transform of $f(t)$, it tells you a lot of information of a specific type about $f(t)$.

27.2. **The pole diagram of the Laplace transform.**

**Summary:** The pole diagram of $F(s)$ tells us a lot about *long-term behavior* of $f(t)$. It tells us *nothing* about the near-term behavior.

This is best seen by examples.

Suppose we have just one pole, at $s = 1$. Among the functions with this pole diagram we have:

$$F(s) = \frac{c}{s-1}\,, \quad G(s) = \frac{ce^{-as}}{s-1}\,, \quad H(s) = \frac{c}{s-1} + b\frac{1-e^{-as}}{s}$$

where $c \neq 0$. (Note that $1 - e^{-as}$ becomes zero when $s = 0$, canceling the zero in the denominator of the second term in $H(s)$.) To be Laplace transforms of real functions we must also assume them all to be real, and $a \geq 0$. Then these are the Laplace transforms of

$$f(s) = ce^t\,, \quad g(t) = \begin{cases} ce^{t-a} & \text{for } t > a, \\ 0 & \text{for } t < a \end{cases}\,, \quad h(t) = \begin{cases} ce^t & \text{for } t > a, \\ ce^t + b & \text{for } t < a \end{cases}$$

All these functions grow like a multiple of $e^t$ when $t$ is large. You can even say which multiple: it is given by the residue at $s = 1$. (Note that $g(t) = (ce^{-a})e^t$, and the residue of $G(s)$ at $s = 1$ is $ce^{-a}$.) But their behavior when $t < a$ is all over the map. In fact, the function can be *anything* for $t < a$, for *any* fixed $a$; as long as it settles down to something close to $ce^t$ for $t$ large, its Laplace transform will have just one pole, at $s = 1$, with residue $c$.

Now suppose we have two poles, say at $s = a + bi$ and $s = a - bi$. Two functions with this pole diagram are

$$F(s) = \frac{c(s-a)}{(s-a)^2 + b^2}\,, \quad G(s) = \frac{cb}{(s-a)^2 + b^2}.$$

and we can modify these as above to find others. These are the Laplace transform of

$$f(t) = ce^{at}\cos(bt)\,, \quad g(t) = ce^{at}\sin(bt).$$

This reveals that it is the *real part* of the pole that determines the long term *growth* of absolute value; if the function oscillates, this means growth of maxima and minima. The *imaginary part* of the pole determines the *angular frequency of oscillation* for large $t$. We can't pick out the phase from the pole diagram alone (but the residues do determine the phase). And we can't promise that it will be exactly sinusoidal times exponential, but it will resemble this. And again, the pole diagram of $F(s)$ says *nothing* about $f(t)$ for small $t$.

Now let's combine several of these, to get a function with several poles. Suppose $F(s)$ has poles at $s = 1$, $s = -1 + i$, and $s = -1 - i$, for example. We should expect that $f(t)$ has a term which grows like $e^t$ (from the pole at $s = 1$), and another term which behaves like $e^{-t}\cos t$ (up to constants and phase shifts). When $t$ is large, the damped oscillation becomes hard to detect as the other term grows exponentially.

We learn that the *rightmost poles dominate*—the ones with *largest real part* have the dominant influence on the long-term behavior of $f(t)$.

The most important consequence relates to the question of *stability*:

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| If all the poles of $F(s)$ have *negative real part* then $f(t)$ decays exponentially to zero as $t \to \infty$. |

If some pole has positive real part, then $|f(t)|$ becomes arbitrarily large for large $t$.

If there are poles on the imaginary axis, and no poles to the right, then the function $f(t)$ may grow (e.g. $f(t) = t$ has $F(s) = 1/s^2$, which has a pole at $s = 0$), but only "sub-exponentially": for any $a > 0$ there is a constant $c$ such that $|f(t)| < ce^{at}$ for all $t > 0$.

**Comment on reality.** We have happily taken the Laplace transform of complex valued functions of $t$: $e^{it} \rightsquigarrow 1/(s-i)$, for example. If $f(t)$ is real, however, then $F(s)$ enjoys a symmetry with respect to complex conjugation:

(2) $$\boxed{\text{If } f(t) \text{ is real-valued then } F(\overline{s}) = \overline{F(s)}.}$$

The pole diagram of a function $F(s)$ such that $F(\overline{s}) = \overline{F(s)}$ is *symmetric about the real axis*: non-real poles occur in complex conjugate pairs. In particular, the pole diagram of the Laplace transform of a real function is symmetric across the real axis.

27.3. **Laplace transform and Fourier series.** We now have two ways to study periodic functions $f(t)$. First, we can form the Laplace transform $F(s)$ of $f(t)$ (regarded as defined only for $t > 0$). Since $f(t)$ is periodic, the poles of $F(s)$ lie entirely along the imaginary axis, and the locations of these poles reveal sinusoidal constituents in $f(t)$, in some sense. On the other hand, $f(t)$ has a Fourier series, which explicitly expresses it as a sum of sinusoidal components. What is the relation between these two perspectives?

For example, the standard square wave $\mathrm{sq}(t)$ of period $2\pi$, with value 1 for $0 < t < \pi$ and $-1$ for $-\pi < t < 0$, restricted to $t > 0$, can be written as

$$\mathrm{sq}(t) = 2(u(t) - u(t-\pi) + u(t-2\pi) - u(t-3\pi) + \cdots) - u(t)$$

By the $t$-shift formula and $u(t) \rightsquigarrow 1/s$,

$$\mathrm{Sq}(s) = \frac{1}{s}\left(2(1 - e^{-\pi s} + e^{-2\pi s} - \cdots) - 1\right) = \frac{1}{s}\left(\frac{2}{1+e^{-\pi s}} - 1\right)$$

The denominator vanishes when $e^{-\pi s} = -1$, and this happens exactly when $s = ki$ where $k$ is an odd integer. So the poles of $\mathrm{Sq}(s)$ are at 0 and the points $ki$ where $k$ runs through odd integers. $s = 0$ does not occur as a pole, because the expression $\dfrac{2}{1+e^{-\pi s}} - 1$ vanishes when $s = 0$ and cancels the $1/s$.

On the other hand, the Fourier series for the square wave is

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right).$$

If we express this as a series of complex exponentials, following 20.6, we find that $c_k$ is nonzero for $k$ an odd integer, positive or negative. There must be a relation!

It is easy to see the connection in general, especially if we use the complex form of the Fourier series,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}.$$

Simply apply the Laplace transform to this expression, using $e^{int} \rightsquigarrow \dfrac{1}{s - in}$:

$$F(s) = \sum_{n=-\infty}^{\infty} \frac{c_n}{s - in}$$

The only possible poles are at the complex numbers $s = in$, and the residue at $in$ is $c_n$.

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| If $f(t)$ is periodic of period $2\pi$, the poles of $F(s)$ occur only at points of the form $n\pi i$ for $n$ an integer, and the residue at $s = n\pi i$ is precisely the complex Fourier coefficients $c_n$ of $f(t)$. |