26. The Laplace transform and generalized functions

The Laplace transform is treated in textbooks as an algorithm for solving initial value problems. This is very misleading. The Laplace transform is much more wonderful than that. It *transforms* one description of some part of the world—given by a signal $f(t)$—to a different desription of the same thing. The new description is again a function, $F(s)$, but now s is a *complex number*. The new description looks nothing like the first one, but certain things are much easier to see from $F(s)$ than from $f(t)$.

The definition, as an integral, shows that in order to compute $F(s)$ for any single value of s, you need to know (essentially) the complete function $f(t)$. It's like a hologram. You've seen this kind of thing before: each Fourier coefficient is an integral involving the whole function. The sequence of Fourier coefficients provides an alternative way of understanding a periodic function, and the Laplace transform will do the same for general functions $f(t)$. The Laplace transform applies to non-periodic functions; but instead it depends only on the values of $f(t)$ for $t \geq 0$.

26.1. **The Laplace transform integral.**

In the integral defining the Laplace transform,

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \,, \tag{1}$$

we really should let s be complex. We are thus integrating a complex-valued function of a real parameter t, $e^{-st}f(t)$, and this is done by integrating the real and imaginary parts separately. It is an improper integral, computed as the limit of $\int_0^T e^{-st} f(t)\, dt$ as $T \to \infty$. (Actually, we will see in Section 26 that it's better to think of the lower limit as "improper" as well, in the sense that we form the integral with lower limit $a < 0$ and then let $a \uparrow 0$.) The textbook assumption that $f(t)$ is of "exponential order" is designed so that if s has large enough real part, the term e^{-st} will be so small (at least for large t) that the product $e^{-st}f(t)$ has an integral which converges as $T \to \infty$. In terms of the pole diagram, we may say that the integral converges when the real part of s is bigger than the real part of any pole in the resulting transform function $F(s)$. The exponential order assumption is designed to guarantee that we won't get poles with arbitrarily large real part.

The region to the right of the rightmost pole is called the **region of convergence**. Engineers abbreviate this and call it the "ROC."

Once the integral has been computed, the expression in terms of s will have meaning for all complex numbers s (though it may take on the value ∞ at some).

For example, let's consider the time-function $f(t) = 1$, $t > 0$. Then:

$$F(s) = \int_0^\infty e^{-st} dt = \lim_{T\to\infty} \left. \frac{e^{-st}}{-s} \right|_0^T = \frac{1}{-s} \left(\lim_{T\to\infty} e^{-sT} - 1 \right).$$

Since $\left| e^{-sT} \right| = e^{-aT}$ if $s = a + bi$, the limit is 0 if $a > 0$ and doesn't exist if $a < 0$. If $a = 0$, $e^{-sT} = \cos(bT) - i \sin(bT)$, which does not have a limit as $T \to \infty$ unless $b = 0$ (which case is not relevant to us since we certainly must have $s \neq 0$). Thus the improper integral converges exactly when $\mathrm{Re}\,(s) > 0$, and gives $F(s) = 1/s$. Despite the fact that the integral definitely diverges for $\mathrm{Re}\,(s) \leq 0$, the expression $1/s$ makes sense for all $s \in \mathbb{C}$ (except for $s = 0$), and it's better to think of the function $F(s)$ as defined everywhere in this way. This process is called " analytic continuation."

26.2. **The lower limit and the t-derivative rule.** Prominent among the signals we want to be able to treat is the delta function $\delta(t)$. What are we to make of the integral (1) in that case? The integrand is $e^{-st}\delta(t)$, which is just $\delta(t)$ since for any s we have $e^{s0} = 1$. The anti-derivative of $\delta(t)$ is the step function $u(t)$, but $u(t)$ doesn't have a well-defined value at $t = 0$. In order to make sense of the integral we have to decide whether the lower limit is actually $0-$ or $0+$; just 0 won't do.

The correct thing to do is to define

$$\boxed{F(s) = \int_{0-}^\infty e^{-st} f(t)\, dt} \tag{2}$$

so that the Laplace transform of $\delta(t)$ is the constant function 1.

In application of the Laplace transform to understanding differential equations, it is important to know what the Laplace transform of $f'(t)$ is. We can calculate this using integration by parts.

$$f'(t) \rightsquigarrow \int_{0-}^\infty e^{-st} f'(t)\, dt = \left. e^{-st} f(t) \right|_{0-}^\infty + s \int_{0-}^\infty e^{-st} f(t)\, dt$$

For s in the region of convergence,

$$e^{-st}f(t)\Big|_{0-}^{\infty} = \lim_{t\uparrow\infty} e^{-st}f(t) - \lim_{t\uparrow 0} e^{-st}f(t) = 0 - f(0-)$$

Therefore we have the t**-derivative rule**

$$\boxed{f'(t) \rightsquigarrow sF(s) - f(0-)} \tag{3}$$

Provided we use the *generalized derivative* as explained in Section 21, this calculation continues to work even when $f(t)$ is merely piecewise differentiable, so its derivative contains delta functions.

The formula (3) is similar to the formula found in textbooks, but more precise. Textbooks will replace $f(0-)$ by $f(0)$. This is fine if $f(t)$ is continuous at $t = 0$, but if it is not then the actual value of $f(t)$ at $t = 0$ can be anything, so the textbook formula can't be correct in that case.

It is possible to get confused about the distinction between $u(t)$ and the constant function $c(t) = 1$, in the context of the Laplace transform. Let's see how the t-derivative rule works out in those cases.

Both $u(t)$ and 1 have Laplace transform $1/s$. Apply (3):

$$\delta(t) = u'(t) \rightsquigarrow s \cdot (1/s) - u(0-) = 1 - 0 = 1$$

$$0 = c'(t) \rightsquigarrow s \cdot (1/s) - c(0-) = 1 - 1 = 0$$

Both are perfectly consistent calculations and not in conflict with each other.

This example shows that the Laplace transform of $f(t)$ depends only on the values of $f(t)$ for $t \geq 0$, but determining the Laplace transform of $f'(t)$ depends also on the value of $f(0-)$. This is because the value of $f(0-)$ is needed to determine whether there is a delta function at $t = 0$ in the (generalized, of course) derivative of $f(t)$.

If we apply (3) to $g(t) = f'(t)$ we get

$$f''(t) = g'(t) \rightsquigarrow sG(s) - g(0-) = s(sF(s) - f(0-)) - f'(0-)\,.$$

This can be continued. The result is particularly simple in case $f^{(k)}(0-) = 0$ for all $k < n$: in that case,

$$f^{(n)}(t) \rightsquigarrow s^n F(s) \tag{4}$$

26.3. **Laplace transform, weight function, transfer function.** Most of the time, Laplace transform methods are inferior to the exponential response formula, undertermined coefficients, and so on, as a way to solve a differential equation. In one specific situation it is quite useful, however, and that is in finding the weight function of an LTI system.

So we want to find the solution of $p(D)w = \delta$ with rest initial conditions. If we apply Laplace transform to the equation, and use (4), we find

$$p(s)W(s) = 1$$

That is to say,

$$\boxed{w(t) \rightsquigarrow \frac{1}{p(s)}}$$

Closely related, we can find the unit step response: $p(D)w_1 = u(t)$ with rest initial conditions gives us $p(D)W_1(s) = 1/s$, or

$$\boxed{w_1(t) \rightsquigarrow \frac{1}{sp(s)}}$$

Example 26.3.1. Suppose the operator is $D^2 + 2D + 2$. The transfer function is $W(s) = 1/(s^2 + 2s + 2) = 1/((s + 1)^2 + 1)$. By the s shift rule and the tables,

$$w(t) = u(t)e^{-t} \sin t\,.$$

The Laplace transform of the unit step response is $W_1(s) = 1/s(s^2 + 2s + 2)$, which we can handle using complex cover up: write

$$\frac{1}{s((s+1)^2+1)} = \frac{a}{s} + \frac{b(s+1)+c}{(s+1)^2+1}\,.$$

Multiply through by s and set $s = 0$ to see $a = 1/2$. Then multiply through by $(s+1)^2+1$ and set $s = -1+i$ to see $bi+c = 1/(-1+i) = (-1-i)/2$, or $b = c = -1/2$: so

$$W_1(s) = \frac{1}{2}\left(\frac{1}{s} - \frac{(s+1)+1}{(s+1)^2+1}\right)\,.$$

Thus the unit step response is

$$w_1(t) = \frac{u(t)}{2}(1 - e^{-t}(\cos t + \sin t))\,.$$

26.4. **What the Laplace transform doesn't tell us.** The Laplace transform is defined by means of an integral. We don't need complete information about a function to determine its integral, so knowing its integral or integrals of products of it with exponentials won't be enough to completely determine it.

For example, if we can integrate a function $g(t)$ then we can also integrate any function which agrees with $g(t)$ except at one value of t, or even except at a finite number of values, and the integral of the new function is the same as the integral of g. Changing a few values doesn't change the "area under the graph."

Thus if $f(t) \rightsquigarrow F(s)$, and $g(t)$ coincides with $f(t)$ except at a few values of t, then also $g(t) \rightsquigarrow F(s)$. We can't hope to recover every value of $f(t)$ from $F(s)$ unless we put some side conditions on $f(t)$, such as requiring that it should be continuous.

Therefore, in working with functions via Laplace transform, when we talk about a function $f(t)$ it is often not meaningful to speak of the value of f at any specific point $t = a$. It does make sense to talk about $f(a-)$ and $f(a+)$, however. Recall that these are defined as

$$f(a-) = \lim_{t \uparrow a} f(t), \qquad f(a+) = \lim_{t \downarrow a} f(t).$$

This means that $f(a-)$ is the limiting value of $f(t)$ as t increases towards a from below, and $f(a+)$ is the limiting value of $f(t)$ as t decreases towards a from above. In both cases, the limit polls infinitely many values of f near a, and isn't changed by altering any finite number of them or by altering $f(a)$ itself; in fact f does not even need to be defined at a for us to be speak of $f(a\pm)$. The best policy is to speak of $f(a)$ only in case both $f(a-)$ and $f(a+)$ are defined and are equal to each other. In this case we can define $f(a)$ to be this common value, and then $f(t)$ is continuous at $t = a$.

The uniqueness theorem for the inverse Laplace transform asserts that if f and g have the same Laplace transform, then $f(a-) = g(a-)$ and $f(a+) = g(a+)$ for all a. If $f(t)$ and $g(t)$ are both continuous at a, so that $f(a-) = f(a+) = f(a)$ and $g(a-) = g(a+) = g(a)$, then it follows that $f(a) = g(a)$.

Part of the strength of the theory of the Laplace transform is its ability to deal smoothly with things like the delta function. In fact, we can form the Laplace transform of a generalized function as described in Section 21, assuming that it is of exponential type. The Laplace transform $F(s)$ determines the singular part of $f(t)$: if $F(s) = G(s)$ then $f_s(t) = g_s(t)$.