23. Impulse and step responses

In real life, we often do not know the parameters of a system (e.g. the spring constant, the mass, and the damping constant, in a spring-mass-dashpot system) in advance. We may not even know the order of the system—there may be many interconnected springs (or diodes). (We will, however, suppose that all the systems we consider are linear and time independent, LTI.) Instead, we often learn about a system by watching how it responds to various input signals.

The simpler the signal, the clearer we should expect the signature of the system parameters to be, and the easier it should be to predict how the system will respond to other more complicated signals. To simplify things we will always begin the system from "rest initial conditions," so x and its derivatives have value zero at $t = 0-$.

In section we will study the response of a system from rest initial conditions to two standard and very simple signals: the unit impulse $\delta(t)$ and the unit step function $u(t)$.

The theory of the convolution integral, Section 24, gives a method of determining the response of a system to *any* input signal, given its unit impulse response.

23.1. **Impulse response.** Given an LTI differential operator $p(D)$, the **unit impulse response** or **weight function** $w(t)$ is the solution to the equation

$$p(D)w = \delta(t) \tag{1}$$

subject to rest initial conditions. Thus always $w(t) = 0$ for $t < 0$.

In the first order case, $a\dot{w} + bw = \delta(t)$ with pre-initial condition $w(0-) = 0$ is equivalent to the post-initial value problem

$$a\dot{w} + bw = 0 \quad , \quad w(0+) = 1/a$$

(See Section 22.) The weight function is thus

$$w(t) = \begin{cases} 0 & \text{for } t < 0 \\ (1/a)e^{-bt/a} & \text{for } t > 0. \end{cases}$$

Note that at $t = 0$ the value jumps by $1/a$, so the derivative contains $(1/a)\delta(t)$ as its singular part. When multiplied by a in the expression $a\dot{w} + bw$, it produces the delta function on the right hand side.

In the second order case, we might have a spring system which is kicked. For example, suppose that the mass is 2, damping constant 4, and spring constant 10, so that the system is governed by the differential operator $p(D) = 2D^2 + 4D + 10I$. Suppose that the system is at rest at $t = 0$, and at that time we kick the mass, providing it with an impulse of 1 unit. This can be modeled by

$$2\ddot{x} + 4\dot{x} + 10x = \delta(t) \quad , \quad x(0-) = 0\,,\ \dot{x}(0-) = 0\,.$$

The impulse changes the momentum by 1, and hence changes the velocity by 1/2, so the equivalent post-initial value problem is

$$2\ddot{x} + 4\dot{x} + 10x = \delta(t) \quad , \quad x(0+) = 0\,,\ \dot{x}(0+) = 1/2\,.$$

For $t > 0$ this equation is homogeneous. The homogeneous equation $2\ddot{x} + 4\dot{x} + 10x = 0$ has an independent set of real solutions is $\{e^{-t}\cos(2t),\ e^{-t}\sin(2t)\}$. The post-initial condition $x(0+) = 0$ forces the weight function to be a multiple of the second homogenous solution, and solving for the coefficient gives $(1/4)e^{-t}\sin(2t)$. Thus

$$w(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ \frac{1}{4}e^{-t}\sin(2t) & \text{for} \quad t > 0. \end{cases}$$

Weight functions of second order equations are continuous, but their derivative jumps at $t = 0$.

The unit impulse response needs to be defined in two parts; it's zero for $t < 0$. This is a characteristic of *causal* systems: the impulse at $t = 0$ has no effect on the system when $t < 0$. In a causal system the unit impulse response is always zero for negative time.

23.2. **Step response.** This is the response of a system at rest to a constant input signal being turned on at $t = 0$. I will write $w_1(t)$ for this system response. If the system is represented by the LTI operator $p(D)$, then $w_1(t)$ is the solution to $p(D)x = u(t)$ with rest initial conditions, where $u(t)$ is the unit step function.

The unit step response can be related to the unit impulse response using (13.2): $Dp(D) = p(D)D$. Using this we can differentiate the equation $p(D)w_1 = 1$ to find that $p(D)(Dw_1) = \delta(t)$, with rest initial conditions. That is to say, $\dot{w}_1(t) = w(t)$, or:

The derivative of the unit step response is the unit impulse response.

If we return to the system represented by $2\ddot{x} + 4\dot{x} + 10x$ considered above, a particular solution to $2\ddot{x}+4\dot{x}+10x = 1$ is given by $x = 1/10$, so the general solution is $x = (1/10) + e^{-t}\,(a\cos(2t) + b\sin(2t))$. Setting

$x(0) = 0$ and $\dot{x}(0) = 0$ leads to

$$w_1(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ \frac{1}{10}\left(1 - e^{-t}(\cos(2t) + \frac{1}{2}\sin(2t))\right) & \text{for} \quad t > 0 \end{cases}$$

You can check that the derivative of this function is $w(t)$ as calculated above. In this example the unit impulse response is a simpler function than the unit step response, and this is generally the case.