22. Generalized functions and differential equations

Now that we know how to express the derivative of a function with a gap in its graph (using the delta function), we can incorporate discontinuous functions and even generalized functions into our study of differential equations.

Suppose we are in the bank account example (2.1) in which $x(t)$ denotes the balance in my bank account at time t. The differential equation modeling my account is

$$\dot{x} - Ix = q(t)$$

where $q(t)$ is the rate of deposit.

Suppose that in the runup to $t = 0$ I don't make any deposits or withdrawals: so the balance as a function of time is given by $x(t) = ce^{It}$. Suppose that the interest rate is 5% per year, so $x(t) = ce^{(0.05)t}$ kilodollars. Suppose $c = 3$.

At time $t = 0$, I deposit two kilodollars, and I wish to model the balance going forward using a differential equation. The *rate* of deposit is thus $q(t) = 2\delta(t)$. This serves as the input signal. What should I take for initial condition? The value of $x(t)$ at $t = 0$ really isn't well defined.

The relevant "initial condition" in this setting is really not $x(0)$, but rather

$$x(0-) = \lim_{t \uparrow 0} x(t) = 3\,.$$

We call this the **pre-initial** condition. After the deposit of two kilodollars at $t = 0$ my balance is five kilodollars, so $x(0+) = 5$.

The natural way to express this scenario as a differential equation is as the "pre-initial value problem"

$$\dot{x} - (0.05)x = 2\delta(t) \quad , \quad x(0-) = 3\,.$$

This is equivalent to the "post-initial value problem"

$$\dot{x} - (0.05)x = 0 \quad , \quad x(0+) = 5\,,$$

which is easy to solve:

$$x(t) = 5e^{(0.05)t} \quad \text{for } t > 0\,.$$

For another example, suppose we have a spring with spring constant 8 and negligible damping. Attached to it is a cart of mass 2 units. So the system is modeled by the simple harmonic oscillator $2\ddot{x} + 8x = 0$. The general solution is the sinusoid of angular frequency 2. Suppose that for $t < 0$ the position x is given by $x(t) = 3\cos(2t)$. At $t = 0$ we supply an impulse of 4 units to the mass, and we are interested in the behavior of the system for $t > 0$.

This scenario is most naturally modeled by the pre-initial value problem

$$2\ddot{x} + 8x = 4\delta(t) \quad , \quad x(0-) = 3 \, , \, \dot{x}(0-) = 0 \, .$$

We can find an equivalent post-initial value problem. The impulse will instantaneously increase the momentum by 4 units, and since momentum is mass times velocity it will increase the velocity $\dot{x}$ by 2 units. No matter how hard I kick the cart, the position won't change instantaneously, however, so in terms of post-initial conditions we have

$$2\ddot{x} + 8x = 0 \quad , \quad x(0+) = 3 \, , \, \dot{x}(0+) = 2 \, .$$

We can go ahead and solve this, too: the solution (for $t > 0$) will be a sinusoid of angular frequency 2. The post-initial conditions give

$$x(t) = 3\cos(2t) + \sin(2t) \, .$$

Here's the general picture.

In a general LTI situation $p(D)x = f(t)$, where the characteristic polyomial $p(s) = a_n s^n + \cdots + a_0$ has degree n (so that $a_n \neq 0$) giving pre-initial conditions means specifying $x(0-)$, $\dot{x}(0-)$, $\ldots$, and $x^{(n-1)}(0-)$. If $f(t)$ has no singularity at $t = 0$, all the functions $x(t), \dot{x}(t), \ldots, x^{(n-1)}(t)$ will be continuous at $t = 0$ and so the pre- and post-initial conditions coincide. If $f(t)$ has $c\delta(t)$ as its singular part at $t = 0$, so that $f(t) = c\delta(t) + g(t)$ where $g(t)$ does not have a singularity at $t = 0$, the equivalent post-initial value problem is given by

$$p(D)x = g(t)$$

subject to the conditions

$$x(0+) = x(0-) \quad , \quad \dot{x}(0+) = x(0-) \quad , \quad \ldots$$
$$x^{(n-1)}(0+) = x^{(n-1)}(0-) + (c/a_n) \, .$$

If all the pre-initial values are zero, we say that we have imposed **rest initial conditions**. So, with the notations as above, the equation $p(D)x = f(t)$ with rest initial conditions is equivalent to the post-initial value problem

$$p(D)x = g(t) \, , \, x(0+) = \cdots = x^{(n-2)}(0+) = 0, x^{(n-1)}(0+) = c/a_n \, .$$