21. Steps, impulses, and generalized functions

In calculus you learn how to model varying quantities using functions. Functions have their limitations, though. By themselves, they are not convenient for modeling some important processes and events, especially those involving sudden changes. In this section we explain how the function concept can be extended to model such processes.

21.1. **Turning on a light:** $u(t)$**.**

When we flick the switch, the light goes on. It seems instantaneous (if it's an old fashioned incandescent bulb, at least). It might be modeled by the **Heaviside step function**

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

For $t < 0$, it's off; for $t > 0$, it's on.

But of course nothing in life is really instantaneous. The current starts to flow, the filament starts to glow. A high-speed camera would show this. It's a matter of time-scale: the time-scale of the lighting-up process is much shorter than the time-scale that our eye can detect. It's a good idea to think of the step-function as a nice *differentiable* function, but one which switches value from 0 to 1 over a time-scale much smaller than the one we are working with.

A consequence of this perspective is that you should not assign any value for $u(0)$. Depending on exactly what happens when the light goes on, it could be anything at $t = 0$. It just doesn't matter. We will leave the value of $u(0)$ *undefined*. Nothing will depend upon it.

21.2. **Home run:** $\delta(t)$**.**

We all know the satisfying "crack" of bat against ball. It looks instantaneous: the ball bounces off the bat and heads into the bleachers. But of course it's not instantaneous. The ball remains in contact with the bat for about one millisecond, during which it deforms quite dramatically and experiences very high (but not infinite!) acceleration. If the ball arrives at 40 m/sec and departs in the opposite direction at the same speed, the velocity changes by 80 m/sec in the course of one millisecond. If this happens at $t = 0$, the velocity can be modeled using the step function as

$$v(t) = 40(2u(t) - 1)\,.$$

The behavior in the millisecond around $t = 0$ may be very interesting, but it's not relevant to modeling what we see without slow motion cameras.

The word "impulse" is used in physics for a change in momentum, usually abrupt. A baseball weighs about .143 kg, so the interaction with the bat provides an impulse of $.143 \times 80$ kg-m/sec to the ball.

Over that millisecond, the ball experiences very high acceleration. The acceleration grows from zero (before the impact) to a very high value and then tapers off again to zero (when the ball leaves the bat). Without much more investigation we don't know very much about the shape of the acceleration graph. But the details don't matter to us; all that matters is that the area under that graph, the integral, is the change in velocity, i.e. 80 (measured in m/sec).

We modeled the velocity using the Heaviside step function. Its derivative, the acceleration, can be modeled too, in terms of the **Dirac delta function** $\delta(t)$. The delta function can be defined as the derivative of the step function,

$$\delta(t) = \frac{du}{dt} \,. \tag{1}$$

We are thinking of $u(t)$ as a smooth function which transitions from value 0 to value 1 very quickly. So its derivative will have value zero except very near to $t = 0$, but in a small interval near zero $\delta(t)$ grows to very large values and then falls again. It may even become negative somewhere in that small interval; all that matters to us is that its *integral* equals 1.

The derivative of $v(t) = 40(2u(t) - 1)$ is $a(t) = 40(2\delta(t)) = 80\delta(t)$; this represents the acceleration experienced by the ball on contact with the bat.

21.3. **Bank deposits:** $x(a+), x(a-)$**.** The step function and the delta function help us model a bank account (2.1). The reality is that I don't get paid continuously; I get paid a lump sum (say two kilodollars, for the sake of argument) at the start of every month, starting, say, at $t = 0$.

Suppose I have ten kilodollars in the bank just before $t = 0$. What is my balance $x(t)$ at $t = 0$? It seems to have two values, differing by two kilodollars. Or, better, its value at exactly $t = 0$ is undefined; what we know is the value just before $t = 0$ (namely, 10) and the value just after $t = 0$ (namely, 12).

There is notation to handle this. For any time a, we write $x(a-)$ for the balance at $t = 0$ as estimated from knowledge of the balance just before $t = a$; mathematically,

(2) $$x(a-) = \lim_{t \uparrow a} x(t).$$

Similarly, $x(a+)$ is the balance at $t = a$ from the perspective of later times; mathematically,

(3) $$x(a+) = \lim_{t \downarrow a} x(t).$$

The actual value we assign as $x(a)$ is unimportant and can be left undeclared. In my bank account, $x(0-) = 10, x(0+) = 12$.

21.4. **Bank deposits: rate and cumulative total.** In (2.1) deposits were recorded using the *rate* of deposit (minus the rate of withdrawal), which we denoted by $q(t)$. We will do that here too, but first think of the *cumulative total* deposit. Write $Q(t)$ for this function. The rate of deposit is the derivative of the cumulative total.

We have to decide on a starting point for $Q(t)$; say $t = 0$. We have to be a little careful, though; I can't really give meaning to $Q(0)$ because I get paid two kilodollars at $t = 0$. What we want is to start *just before* $t = 0$: so

$$Q'(t) = q(t) \quad , \quad Q(0-) = 0\,.$$

Now we can model my cumulative total paycheck deposits using shifts of the step function:

$$Q(t) = 2u(t) + 2u(t - \tfrac{1}{12}) + 2u(t - \tfrac{2}{12}) + \cdots\,.$$

Sketch a graph of this function!

Using shifts of the delta function, we can also write a formula for the rate of paycheck deposit:

$$q(t) = 2\delta(t) + 2\delta(t - \tfrac{1}{12}) + 2\delta(t - \tfrac{2}{12}) + \cdot\,.$$

21.5. **Generalized functions.** When these shifted and scaled delta functions are added to "ordinary" functions you get a "generalized function." I'll describe a little part of the theory of generalized functions. The next few paragraphs will sound technical. I hope they don't obscure the simplicity of the idea of generalized functions as a model for abrupt changes.

I will use the following extensions of a definition from Edwards and Penney (p. 271 in the sixth edition). To prepare for it let me call a collection of real numbers $a_1, a_2, \ldots$, **sparse** if for any $r > 0$ there are only finitely many of k such that $|a_k| < r$. So any finite collection of numbers is sparse; the collection of whole numbers is sparse; but the collection of numbers $1, 1/2, 1/3, \ldots$, is not sparse. Sparse sets don't bunch up. The empty set is sparse.

When I describe a function (on an interval) I typically won't insist on knowing its values for *all* points in the interval. I'll allow a sparse collection of points at which the value is undefined. We already saw this in the definition of $u(t)$ above.

A function $f(t)$ (on an interval) is **piecewise continuous** if (1) it is continuous everywhere (in its interval of definition) except at a sparse collection of points; and (2) for every a, both $f(a+)$ and $f(a-)$ exist. (They are equal exactly when $f(t)$ is continuous at $t = a$.)

A function $f(t)$ is **piecewise differentiable** if (1) it is piecewise continuous, (2) it is differentiable everywhere except at a sparse collection of points, and its derivative is piecewise continuous.

We now want to extend this by including delta functions. A **generalized function** is a piecewise continuous function $f_r(t)$ plus a linear combination of delta functions,

$$f_s(t) = \sum_k b_k \delta(t - a_k), \tag{4}$$

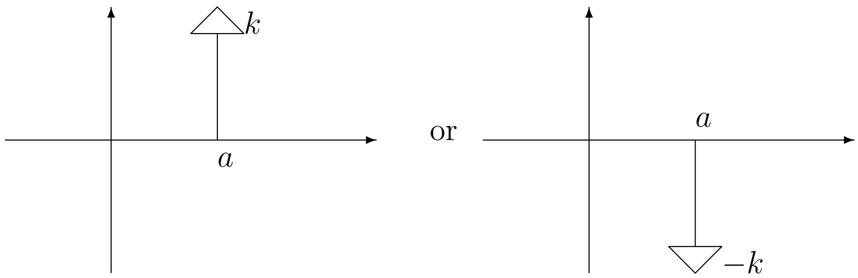
where the a_k's form a sparse set.

Write $f(t)$ for the sum:

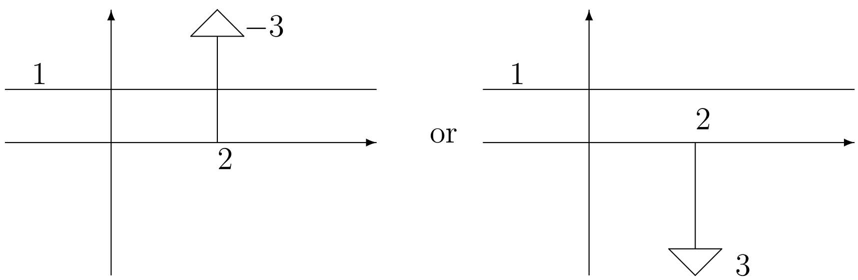
$$f(t) = f_r(t) + f_s(t).$$

$f_r(t)$ is the "regular part" of $f(t)$, and $f_s(t)$ is the "singular part." We define $f(a-)$ to be $f_r(a-)$ and $f(a+)$ to be $f_r(a+)$. Since the actual value of $f(t)$ at $t = a$ is not used to compute these limits, this is a good definition even if a is one of the a_k's.

We will use a "harpoon" to denote a delta function in a graph. The harpoon should be thought to be very high, and resting on the horizontal axis. This notation by itself does not include the information of the area under the graph. To deal with this we will decorate the barb of the harpoon representing $k\delta(t - a)$ with the number k. k may be negative, in which case the harpoon might better be thought of as extending downward. We will denote the same function, $k\delta(t - a)$, equally by a downward harpoon decorated with $-k$:



For example, $1 - 3\delta(t-2)$ can be denoted by either of the following graphs.



A harpoon with $k = 0$ is the same thing as no harpoon at all: $0\delta(t-a) = 0$. We'll call the term $b_k\delta(t-a_k)$ occurring in $f_s(t)$ **the singularity of** $f(t)$ **at** $t = a_k$. If a is not among the a_k's (or if $a = a_k$ but $b_k = 0$) then there is no singularity in $f(t)$ at $t = a$.

21.6. **Integrating generalized functions.** Generalized functions are set up so they can be integrated. $\delta(t-a) = u'(t-a)$, so by the fundamental theorem of calculus

$$\int_b^c \delta(t-a)\,dt = u(c-a) - u(b-a).$$

If $b < a < c$, this is 1. If a is not between b and c, this is 0. If $a = b$ or $a = c$ then this integral involves the expression $u(0)$, which is undefined; so the integral is undefined. We can however define

$$\int_{b-}^{c+} f(t)\,dt = \lim_{b'\uparrow b} \lim_{c'\downarrow c} \int_{b'}^{c'} f(t)\,dt,$$

and this gives a well defined result when $f(t) = \delta(t-a)$: Assuming $b \leq c$,

$$\int_{b-}^{c+} \delta(t-a)\,dt = 1 \quad \text{if} \quad b \leq a \leq c,$$

and zero otherwise. In particular,

$$\int_{a-}^{a+} \delta(t-a)\,dt = 1.$$

Now if $f(t)$ is any generalized function, we can define the integral

$$\int_{b-}^{c+} f(t)\,dt$$

by integrating the regular part of $f(t)$ in the usual way, and adding the sum of the b_k's over k for which $b \leq a_k \leq c$ (using the notation of (4)).

21.7. **The generalized derivative.** Generalized functions let us make sense of the derivative of a function which is merely piecewise differentiable.

For example, we began by saying that the "derivative" of the piecewise differentiable function $u(t-a)$ is the generalized function $\delta(t-a)$. This understanding lets us define the **generalized derivative** of any piecewise continuously differentiable function $f(t)$. It is a generalized function and we denote it by $f'(t)$. Its regular part, $(f')_r(t)$, is the usual derivative of $f(t)$ (which is defined except where the graph of $f(t)$ has breaks or corners), and its singular part is given by the sum of terms

$$(f(a+) - f(a-))\delta(t-a),$$

summed over the values a of t where the graph of $f(t)$ has breaks. Each shifted and scaled δ function records the instantaneous velocity needed to accomplish a sudden jump in the value of $f(t)$. When the graph of $f(t)$ has a corner at $t = a$, the graph of $f'(t)$ has a jump at $t = a$ and isn't defined at $t = a$ itself; this is a discontinuity in the piecewise continuous function $(f')_r(t)$.

With this definition, the "fundamental theorem of calculus"

$$\int_{b-}^{c+} f'(t)\,dt = f(c+) - f(b-) \tag{5}$$

holds for generalized functions.

For further material on this approach to generalized functions the reader may consult the article "Initial conditions, generalized functions, and the Laplace transform," IEEE Control Systems Magazine 27 (2007) 22–35, by Kent Lundberg, Haynes Miller, and David Trumper. `http://math.mit.edu/~hrm/papers/lmt.pdf`.