18. Linearization: the phugoid equation as example

"Linearization" is one of the most important and widely used mathematical terms in applications to Science and Engineering. In the context of Differential Equations, the word has two somewhat different meanings.

On the one hand, it may refer to the procedure of analyzing solutions of a nonlinear differential equation near a critical point by studying an approximating linear equation. This is linearizing an *equation*.

On the other hand, it may refer to the process of systematically dropping negligibly small terms in the mathematical expression of the model itself, under the assumption that one is near an equilibrium. The result is that you obtain a linear differential equation directly, without passing through a nonlinear differential equation. This is linearizing a *model*.

A virtue of the second process is that it avoids the need to work out the full nonlinear equation. This may be a challenging problem, often requiring clever changes of coordinates; while, in contrast, it is always quite straightforward to write down the linearization near equilibrium, by using a few general ideas. We will describe some of these ideas in this section.

Most of the time, the linearization contains all the information about the behavior of the system near equilibrium, and we have a pretty complete understanding of how linear systems behave, at least in two dimensions. There aren't too many behaviors possible. The questions to ask are: is the system stable or unstable? If it's stable, is it underdamped (so the solution spirals towards the critical point) or overdamped (so it decays exponentially without oscillation)? If it's underdamped, what is the period of oscillation? In either case, what is the damping ratio?

One textbook example of this process is the analysis of the linear pendulum. In this section we will describe a slightly more complicated example, the "phugoid equation" of airfoil flight.

18.1. **The airplane system near equilibrium.** If you have ever flown a light aircraft, you know about "dolphining" or "phugoid oscillation." This is precisely the return of the aircraft to the equilibrium state of steady horizontal flight. We'll analyze this effect by linearizing the model near to this equilibrium. To repeat, the questions to ask are: Is this equilibrium stable or unstable? (Experience suggests it's stable!) Is it overdamped or underdamped? What is the damping ratio? If it's underdamped, what is the period (or, more properly, the quasiperiod)?

There are four forces at work: thrust F, lift L, drag D, and weight $W = mg$. At equilibrium the forces cancel. Here's a diagram. In it the airplane is aligned with the thrust vector, since the engines provide a force pointing parallel with the body of the airplane.

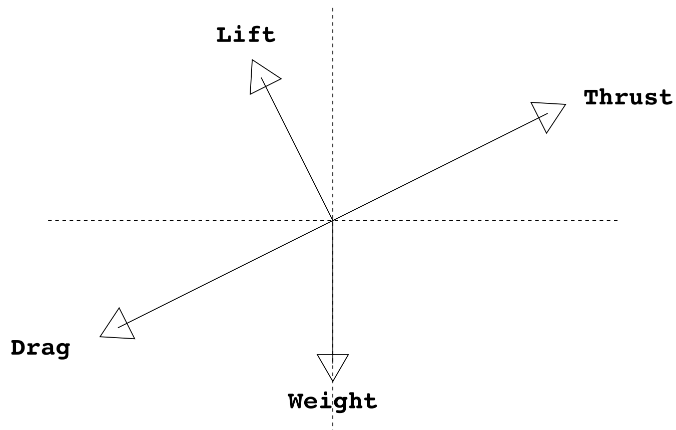


FIGURE 10. Forces on an airfoil

I'll make the following simplifying assumptions: **(1)** the air is still relative to the ground (or, more generally, the ambient air is moving uniformly and we use a coordinate frame moving with the air); **(2)** the weight and the thrust are both constant.

Lift and the drag are more complicated than weight and thrust. They are components of a "frictional" force exerted on the plane by the surrounding air. The drag is, by definition, the component of that force in the direction of the thrust (directed backwards), and the lift is the perpendicular component, directed towards the "up" side of the airfoil.

When we call this force "frictional," what we mean is that it depends upon the velocity of the plane (through the air) and on nothing else.

Friction is a complex process, and it shows up differently in different regimes. Let's first think about friction of a particle moving along the x axis. It is then a force $\phi(v)$ dependent upon $v = \dot{x}$. It always takes the value zero when the velocity is zero and is directed against the direction of motion. The tangent line approximation then lets us approximate $\phi(v)$ by a multiple of v when $|v|$ is small. This is "linear damping," and it plays a big role in our study of second order LTI systems. When the velocity is relatively large, consideration of the nonlinear dependence of friction on velocity becomes unavoidable. Often, for v in a range of values the frictional force is reasonably well approximated by a power law:

$$\phi(v) = \begin{cases} -c|v|^p & \text{for } v \geq 0 \\ c|v|^p & \text{for } v < 0 \end{cases} \tag{1}$$

where $c > 0$ is a constant. This rather complicated looking expression guarantees that the force acts against the direction of motion. The magnitude is $|\phi(v)| = c|v|^p$.

Often the power involved is $p = 2$, so $\phi(v) = -cv^2$ when $v > 0$. (Since squares are automatically positive we can drop the absolute values and the division into cases in (1).) To analyze motion near a given velocity v_0, the tangent line approximation indicates that we need only study the rate of change of $\phi(v)$ near the velocity v_0, and when $p = 2$ and $v_0 > 0$,

$$\phi'(v_0) = -2cv_0 = \frac{2\phi(v_0)}{v_0}\,. \tag{2}$$

We rewrote the derivative in terms of $\phi(v_0)$ because doing so eliminates the constant c.

Now let's go back to the airfoil. Our last assumption is that near equilibrium velocity v_0, drag and lift depend quadratically on speed. Stated in terms of (2) we have our next assumption: **(3)** the drag $D(v)$ and the lift $L(v)$ are quadratic, so by (2) they satisfy

$$D'(v_0) = \frac{2D(v_0)}{v_0}\,, \qquad L'(v_0) = \frac{2L(v_0)}{v_0}\,.$$

There is an equilibrium velocity at which the forces are in balance: cruising velocity v_0. Our final assumption is that at cruising velocity the pitch of the airplane is small: so **(4)** the horizontal component of lift is small. The effect is that to a good approximation, lift balances weight and thrust balances drag:

$$D(v_0) = F\,, \qquad L(v_0) = mg\,.$$

This lets us rewrite the equations for the derivatives can be rewritten

$$D'(v_0) = \frac{2F}{v_0}\,, \qquad L'(v_0) = \frac{2mg}{v_0}\,. \tag{3}$$

This is all we need to know about the dynamics of airfoil flight.

There are several steps in our analysis of this situation from this point. A preliminary observation is that in the phugoid situation the airplane has no contact with the ground, so **everything is invariant under space translation**. After all, the situation is the same for all altitudes (within a range over which atmospheric conditions and gravity are reasonably constant) and for all geographical locations. The implication is that Newton's Law can be written entirely in terms of velocity and its derivative, acceleration. Newton's Law is a *second order* equation for position, but if the forces involved don't depend upon position it can be rewritten as a *first order* equation for velocity. This reasoning is known as *reduction of order*.

18.2. **Deriving the linearized equation of motion.** The fundamental decision of linearization is this:

> Study the situation near the equilibrium we care about, and systematically use the tangent line approximation at that equilibrium to simplify expressions.

The process of replacing a function by its tangent line approximation is referred to as "working to first order."

Let's see how this principle works out in the phugoid situation.

One of the first steps in any mathematical analysis is to identify and give symbols for relevant parameters of the system, and perhaps to set up a well-adapted coordinate system. Here, we are certainly interested in the velocity. We have already introduced v_0 for the equilibrium velocity, which by assumption **(4)** is horizontal. We write the actual velocity as equilibrium plus a correction term: Write

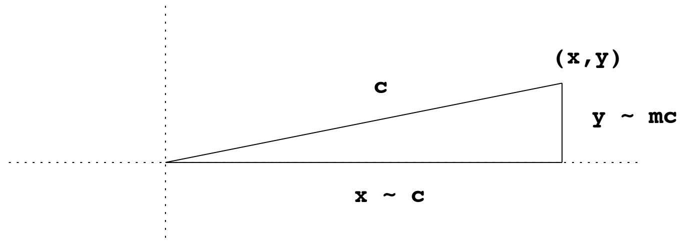
w for the vertical component of velocity, and

$v_0 + u$ for the horizontal component,

and suppose the axes are arranged so that the plane is moving in the direction of the positive x axis. We are assuming that the plane is not too far from equilibrium, so we are assuming that w and u are both small.

We will want to approximate the actual speed in terms of v_0, u, and w. To do this, and for other reasons too, we will use a geometric principle which arises very often in linearization of physical systems.

> If a vector makes a small angle with the positive x axis, then to first order its x component is its length and its y component is its length times the slope.



This is geometrically obvious, and equivalent to the facts that $\cos'(0) = 0$ and $\sin'(0) = 1$.

If we take $x = v_0 + u$, $y = w$, and $c = v$, the estimate $x \sim c$ says that the speed is approximately $v_0 + u$; the normal component w makes only a "second order" contribution and we will ignore it.

Now we use the linearization principle again: we plug this estimate of the speed into the tangent line approximation for $D(v)$ and $L(v)$ and use (3) and the values $D(v_0) = F$ and $L(v_0) = mg$ to find

$$D \simeq F + \frac{2F}{v_0}u\,, \quad L \simeq mg + \frac{2mg}{v_0}u\,.$$

Subscript L, W, T, and D by h and v to denote their horizontal and vertical components. Writing down similar triangles, we find (to first order, always—ignoring terms like u^2, uw, and w^2):

$$L_v \simeq L \simeq mg + \frac{2mg}{v_0}u\,, \quad L_h \simeq \frac{w}{v_0}L \simeq \frac{w}{v_0}mg$$

$$W_v = mg\,, \quad W_h = 0\,, \quad T_v = \frac{w}{v_0}F\,, \quad T_h \simeq F$$

$$D_v \simeq \frac{w}{v_0}D \simeq \frac{w}{v_0}F\,, \quad D_h \simeq D \simeq F + \frac{2F}{v_0}u.$$

In words, to first order the vertical components of thrust and drag still cancel and the vertical component of the lift in excess of the weight is given by $(2mg/v_0)u$, so, by Newton's law,

$$m\dot{w} = \frac{2mg}{v_0}u. \tag{4}$$

Also, to first order, the horizontal component of the excess of drag over thrust is $(2F/v_0)u$, and the horizontal component of the lift is $-mg(w/v_0)$: so

$$m\dot{u} = -\frac{2F}{v_0}u - \frac{mg}{v_0}w. \tag{5}$$

We can package these findings in matrix terms:

$$\frac{d}{dt}\begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} -2F/mv_0 & -g/v_0 \\ 2g/v_0 & 0 \end{bmatrix}\begin{bmatrix} u \\ w \end{bmatrix}. \tag{6}$$

and we could go on to use the methods of linear systems to solve it. Instead, though, we will solve the equations (4), (5) by elimination. Differentiating the equation for $\dot{w}$ and substituting the value for $\dot{u}$ from the other equation gives the homogeneous second order constant coefficient linear differential equation

$$\boxed{\ddot{w} + \frac{2F}{mv_0}\dot{w} + \frac{2g^2}{v_0^2}w = 0} \tag{7}$$

18.3. **Implications.** From this (or from the system (6)) we can read off the essential characteristics of motion near equilibrium. We have in (7) a second order homogeneous linear ODE with constant coefficients; it is of the form

$$\ddot{w} + 2\zeta\omega_n\dot{w} + \omega_n^2 w = 0,$$

where ω_n is the *natural angular frequency* and ζ is the *damping ratio* (for which see Section 15). Comparing coefficients,

$$\omega_n = \frac{\sqrt{2}\,g}{v_0} \quad , \qquad \zeta = \frac{F}{\sqrt{2}\,mg}.$$

We have learned the interesting fact that the period

$$P = \frac{2\pi}{\omega_n} = \frac{\sqrt{2}\,\pi}{g}v_0$$

of phugoid oscillation depends *only on the equilibrium velocity* v_0. In units of meters and seconds, P is about $0.45\,v_0$. The nominal equilibrium speeds v_0 for a Boeing 747 and an F15 are 260 m/sec and 838 m/sec, respectively. The corresponding phugoid periods are about 118 sec and 380 sec.

We have also discovered that the phugoid damping ratio depends *only on the "thrust/weight ratio,"* a standard tabulated index for aircraft. Both ζ and F/mg are dimensionless ratios, and ζ is about $.707(F/mg)$, independent of units. F/mg is about 0.27 for a Boeing 747, and about 0.67 for an F15.

The system is underdamped as long as $\zeta < 1$, i.e. $(F/mg) < \sqrt{2}$. Even an F15 doesn't come close to having a thrust/weight approaching 1.414.

To see a little more detail about these solutions, let's begin by supposing that the damping ratio is negligible. The equation (7) is then simply a harmonic oscillator with angular frequency ω_n, with general solution of the form

$$w = w_0 \cos(\omega_n t - \phi)\,.$$

Equation (4) then shows that $u = (v_0/2g)\dot{w} = -(v_0/2g)\omega_n w_0 \sin(\omega_n t - \phi)$. But $\omega_n = \sqrt{2}g/v_0$, so this is

$$u = -(w_0/\sqrt{2})\sin(\omega_n t - \phi)\,.$$

That is: The vertical amplitude is $\sqrt{2}$ times as great as the horizontal amplitude.

Integrate once more to get the motion in space:

$$x = x_0 + v_0 t + a\cos(\omega_n t - \phi)$$

where $a = v_0 w_0/g$—as a check, note that a does have units of length!—and

$$y = y_0 + \sqrt{2}\,a\,\sin(\omega_n t - \phi)\,,$$

for appropriate constants of integration x_0 (which is the value of x at $t = 0$) and y_0 (which is the average altitude). Relative to the frame of equilibrium motion, the plane executes an ellipse whose vertical axis is $\sqrt{2}$ times its horizontal axis, moving counterclockwise. (Remember, the plane is moving to the right.)

Relative to the frame of the ambient air, the plane follows a roughly sinusoidal path. The horizontal deviation u from equilibrium velocity is small and would be hard to detect in the flightpath.

Reintroducing the damping, the plane spirals back to equilibrium.

We can paraphrase the behavior in physics terms like this: Something jars the airplane off of equilibrium; suppose it is hit by a downdraft and the vertical component of its velocity, w, acquires a negative value. This puts us on the leftmost point on the loop. The result is a decrease in altitude, and the loss in potential energy translates to a gain in kinetic energy. The plane speeds up, increasing the lift, which counteracts the negative w. We are now at the bottom of the loop. The excess velocity continues to produce excess lift, which raises the plane past equilibrium (at the rightmost point on the loop). The plane now has $w > 0$, and rises above its original altitude. Kinetic energy is converted to potential energy, the plane slows down, passes through the top of the loop; the lowered speed results in less lift, and the plane returns to where it was just after the downdraft hit (in the frame of equilibrium motion).

A typical severe downdraft has speed on the order of 15 m/sec, so we might take $c = 10$ m/sec. With the 747 flying at 260 m/sec, this results in a vertical amplitude of 265 meters; the F15 flying at 838 m/sec gives a vertical amplitude of 855 meters, which could pose a problem if you are near the ground!

Historical note: The term *phugoid* was coined by F. W. Lanchester in his 1908 book *Aerodonetics* to refer to the equations of airfoil flight. He based this neologism on the Greek *φυγή*, which does mean flight, but in the sense of the English word *fugitive*, not in the sense of movement through the air. Evidently Greek was not his strong suit.

Question: Assumption **(3)** is the most suspect part of this analysis. Suppose instead of quadratic dependence we assume some other power law, for lift and drag. What is the analogue of (3), and how does this alter our analysis?