17. The Tacoma Narrows bridge

On July 1, 1940, a bridge spanning the Tacoma Narrows opened to great celebration. It dramatically shortened the trip from Seattle to the Kitsap Peninsula. It was an elegant suspension bridge, a mile long (third longest in the US at the time) but just 39 feet across. Through the summer and early fall, drivers noticed that it tended to oscillate vertically, quite dramatically. It came to be known as "Galloping Gertie." "Motorists crossing the bridge sometimes experienced "roller-coaster like" travel as they watched cars ahead almost disappear vertically from sight, then reappear." (Quoted from Billah-Scanlon.)

During the first fall storm, on November 7, 1940, with steady winds above 40 mph, the bridge began to exhibit a different behavior. It *twisted*, part of one edge rising while the opposing edge fell, and then the reverse. At 10:00 AM the bridge was closed. The torsional oscillations continued to grow in amplitude, till, at just after 11:00, the central span of the bridge collapsed and fell into the water below. One car and a dog were lost.

Why did this collapse occur? Were the earlier oscillations a warning sign? Many differential equations textbooks announce that this is an example of *resonance*: the gusts of wind just happened to match the natural frequency of the bridge.

The problem with this explanation is that the wind was not gusting—certainly not at anything like the natural frequency of the bridge. This explanation is worthless.

Structural engineers have studied this question in great detail. They had determined already before the bridge collapsed that the vertical oscillation was self-limiting, and not likely to lead to a problem. The torsional oscillation was different. To model it, pick a portion of the bridge far from the support towers. Let $\theta(t)$ denote its angle off of horizontal, as a function of time. The torsional dynamics can be modeled by a second order differential equation of the form

$$\ddot{\theta} + b_0\dot{\theta} + k_0\theta = F$$

where k_0 is the square of the natural angular frequency of the torsional oscillation and b_0 is a damping term. The forcing term F depends upon θ itself, and its derivatives, and on the wind velocity v. To a reasonable approximation we can write

$$F = -k(v)\theta - b(v)\dot{\theta}$$

where $k(v)$ and $b(v)$ are functions of v which are determined by the bridge characteristics.

This equation can be rewritten as

$$\ddot{\theta} + (b_0 + b(v))\dot{\theta} + (k_0 + k(v))\theta = 0 \tag{1}$$

In our situation, the wind velocity changes slowly relative to the time scale of the oscillation, so this is a second order linear differential equation with constant coefficients in which the damping constant and the spring constant depend upon the wind velocity.

It turns out that in the case of the Tacoma Narrows bridge the value of $k(v)$ is small relative to k_0; the effect is to slightly alter the effective natural frequency of torsional oscillation.

The function $b(v)$ reflects mainly turbulence effects. The technical term for this effect is *flutter*. The same mechanism makes flags flap and snap in the wind. It turns out that the graph of $b(v)$ has a shape somewhat like the curve displayed in Figure 9.

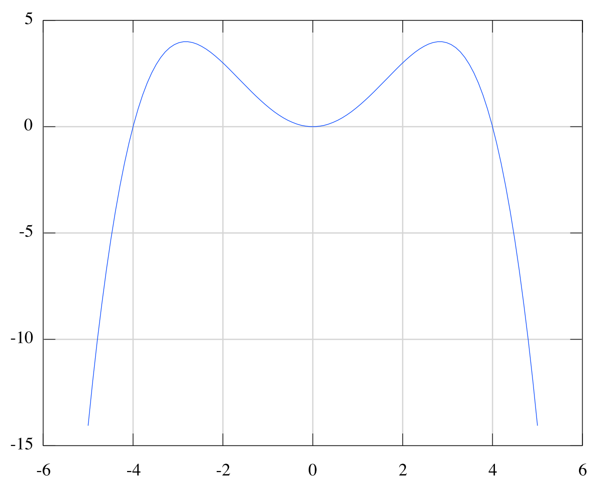

FIGURE 9. $b(v)$

When $|v|$ is small, $b(v) > 0$: the wind actually increases the damping of the bridge; it becomes *more* stable. When $|v|$ is somewhat larger, $b(v) = 0$, and the wind has no damping effect. When $|v|$ increases still more, $b(v)$ becomes negative and it starts to erode the damping of the bridge, till, when it hits a certain critical value, it overwhelms the intrinsic damping of the bridge. The result is *anti-damping*, a negative effective damping constant. For the Tacoma Narrows Bridge, the critical value of velocity was discovered, on that day in November, 1940, to be around 40 miles per hour.

In more detail, solutions to (1) are linear combinations of functions of the form e^{rt} where r is a root of the characteristic polynomial $p(s) = s^2 + (b_0 + b(v))s + (k_0 + k(v))$:

$$r = -\frac{b_0 + b(v)}{2} \pm \sqrt{\frac{(b_0 + b(v))^2}{4} - (k_0 + k(v))}$$

As long as $|b_0 + b(v)|$ isn't too big, the contents of the square root will be negative: the roots have nonzero imaginary parts, indicating oscillation. The real part of each root is $a = -(b_0 + b(v))/2$, which is positive if v is such that $b(v) < -b_0$. If we write $r = a \pm i\omega$, the general solution is

$$\theta = Ae^{at}\cos(\omega t - \phi)$$

Its peaks grow in magnitude, exponentially.

This spells disaster. There are compensating influences which slow down the rate of growth of the maxima, but in the end the system will—and did—break down.

Reference:

K. Y. Billah and R. H. Scanlan, Resonance, Tacoma Narrows bridge failure, and undergraduate physics textbooks, Am. J. Phys. 59 (1991) 118–124.