12. Resonance

12.1. **Resonance.** When you swing your kid sister the key is to get in synch with the natural frequency of the swing. This is called "resonance." We might model the swing-and-sister setup by a sinusoidally driven harmonic oscillator, $\ddot{x} + \omega_n^2 = A\cos(\omega t)$. In (10.9) we saw that this has a periodic solution

$$x_p = A\frac{\cos(\omega t)}{\omega_n^2 - \omega^2} \tag{1}$$

provided that $\omega \neq \omega_n$. Resonance occurs when the two frequencies coincide. The model isn't very accurate; there are *no* bounded solutions to our equation when $\omega_n = \omega$. But we neglected damping. . . .

From a more sophisticated perspective, resonance occurs in the equation $p(D)x = e^{rt}$ when r is a root of the characteristic polynomial $p(s)$; for then the denominator in the Exponential Response Formula vanishes.

This occurs in the complex replacement for the harmonic oscillator, $\ddot{z} + \omega_n^2 z = Ae^{i\omega t}$ when $\omega = \pm\omega_n$, and acconts for the vanishing denominator in (1).

It also occurs if we try to use the ERF to solve $\dot{x} + x = e^{-t}$. The Exponential Response Formula proposes a solution $x_p = e^{-t}/p(-1)$, but $p(-1) = 0$ so this fails. There is no solution of the form ce^{rt}.

Here is a way to solve $p(D)x = e^{rt}$ when this happens. The ERF came from the calculation

$$p(D)e^{rt} = p(r)e^{rt},$$

which is valid whether or not $p(r) = 0$. We will take this expression and *differentiate it with respect to* r, keeping t constant. The result, using the product rule and the fact that partial derivatives commute, is

$$p(D)te^{rt} = p'(r)e^{rt} + p(r)te^{rt}$$

If $p(r) = 0$ this simplifies to

$$p(D)te^{rt} = p'(r)e^{rt}\,. \tag{2}$$

Now if $p'(r) \neq 0$ we can divide through by it and see:

The Resonant Exponential Response Formula: If $p(r) = 0$ then a solution to $p(D)x = ae^{rt}$ is given by

$$\boxed{x_p = a\,\frac{te^{rt}}{p'(r)}} \tag{3}$$

provided that $p'(r) \neq 0$.

In our example above, $p(s) = s + 1$ and $r = 1$, so $p'(r) = 1$ and $x_p = te^{-t}$ is a solution.

This example exhibits a characteristic feature of resonance: the solutions grow faster than you might expect. The characteristic polynomial leads you to expect a solution of the order of e^{-t}. In fact the solution is t times this. It still decays to zero as t grows, but not as fast as e^{-t} does.

Example 12.1.1. Let's return to the harmonic oscillator represented by $\ddot{x} + \omega_n^2 x$, or by the operator $D^2 + \omega_n^2 I = p(D)$, driven by the signal $A\cos(\omega t)$. This ODE is the real part of

$$\ddot{z} + \omega_n^2 z = Ae^{i\omega t}\,,$$

so the Exponential Response Formula gives us the periodic solution

$$z_p = A\frac{e^{i\omega_n t}}{p(i\omega)}\,.$$

This is fine *unless* $\omega = \omega_n$, in which case $p(i\omega_n) = (i\omega_n)^2 + \omega_n^2 = 0$; so the amplitude of the proposed sinusoidal response should be infinite. The fact is that there is *no* periodic system response; the system is in *resonance* with the signal.

To circumvent this problem, let's apply the Resonance Exponential Response Formula: since $p(s) = s^2 + \omega_n^2$, $p'(s) = 2s$ and $p'(i\omega_n) = 2i\omega_0$, so

$$z_p = A\frac{te^{i\omega_n t}}{2i\omega_n}\,.$$

The real part is

$$x_p = \frac{A}{2\omega_n} t\sin(\omega_n t)\,.$$

The general solution is thus

$$x = \frac{A}{2\omega_n} t\sin(\omega_n t) + b\cos(\omega_n t - \phi)\,.$$

In words, all solutions oscillate with pseudoperiod $2\pi/\omega_n$, and grow in amplitude like $At/(2\omega_n)$. When ω_n is large—high frequency—this rate of growth is small.

12.2. **Higher order resonance.** It may happen that both $p(r) = 0$ and $p'(r) = 0$. The general picture is this: Suppose that k is such that $p^{(j)}(r) = 0$ for $j < k$ and $p^{(k)}(r) \neq 0$. Then $p(D)x = ae^{rt}$ has as solution

$$x_p = a\frac{t^k e^{rt}}{p^{(k)}(r)} . \tag{4}$$

For instance, if $\omega = \omega_0 = 0$ in Example 12.1.1, $p'(i\omega) = 0$. The signal is now just the constant function a, and the ODE is $\ddot{x} = a$. Integrating twice gives $x_p = at^2/2$ as a solution, which is a special case of (4), since $e^{rt} = 1$ and $p''(s) = 2$.

You can see (4) in the same way we saw the Resonant Exponential Response Formula. So take (2) and differentiate again with respect to r:

$$p(D)t^2e^{rt} = p''(r)e^{rt} + p'(r)te^{rt}$$

If $p'(r) = 0$, the second term drops out and if we suppose $p''(r) \neq 0$ and divide through by it we get

$$p(D)\left(\frac{t^2e^{rt}}{p'(r)}\right) = e^{rt}$$

which the case $k = 2$ of (4). Continuing, we get to higher values of k as well.