LS.5 Theory of Linear Systems

1. General linear ODE systems and independent solutions.

We have studied the homogeneous system of ODE's with constant coefficients,

$$\mathbf{x}' = A\mathbf{x}\,, \tag{1}$$

where A is an $n \times n$ matrix of constants $(n = 2, 3)$. We described how to calculate the eigenvalues and corresponding eigenvectors for the matrix A, and how to use them to find n independent solutions to the system (1).

With this concrete experience solving low-order systems with constant coefficients, what can be said in general when the coefficients are not constant, but functions of the independent variable t? We can still write the linear system in the matrix form (1), but now the matrix entries will be functions of t:

$$\begin{aligned} x' &= a(t)x + b(t)y \\ y' &= c(t)x + d(t)y \end{aligned}\,, \qquad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{2}$$

or in more abridged notation, valid for $n \times n$ linear homogeneous systems,

$$\mathbf{x}' = A(t)\,\mathbf{x}\,. \tag{3}$$

Note how the matrix becomes a function of t — we call it a "matrix-valued function" of t, since to each value of t the function rule assigns a matrix:

$$t_0 \to A(t_0) = \begin{pmatrix} a(t_0) & b(t_0) \\ c(t_0) & d(t_0) \end{pmatrix}$$

In the rest of this chapter we will often not write the variable t explicitly, but it is always understood that the matrix entries are functions of t.

We will sometimes use $n = 2$ or 3 in the statements and examples in order to simplify the exposition, but the definitions, results, and the arguments which prove them are essentially the same for higher values of n.

Definition 5.1 Solutions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ to (3) are called **linearly dependent** if there are constants c_i, not all of which are 0, such that

$$c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t) = 0, \qquad \text{for all } t. \tag{4}$$

If there is no such relation, i.e., if

$$c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t) = 0 \quad \text{for all } t \quad \Rightarrow \quad \text{all } c_i = 0, \tag{5}$$

the solutions are called **linearly independent**, or simply *independent*.

The phrase "for all t" is often in practice omitted, as being understood. This can lead to ambiguity; to avoid it, we will use the symbol $\equiv 0$ for **identically 0**, meaning: "zero for all t"; the symbol $\not\equiv 0$ means "not identically 0", i.e., there is some t-value for which it is not zero. For example, (4) would be written

$$c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t) \;\equiv\; 0\;.$$

Theorem 5.1 *If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is a linearly independent set of solutions to the $n \times n$ system $\mathbf{x}' = A(t)\mathbf{x}$, then the general solution to the system is*

$$\mathbf{x} \;=\; c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n \tag{6}$$

Such a linearly independent set is called a **fundamental** *set of solutions.*

> This theorem is the reason for expending so much effort in LS.2 and LS.3 on finding two independent solutions, when $n = 2$ and A is a constant matrix. In this chapter, the matrix A is not constant; nevertheless, (6) is still true.

Proof. There are two things to prove:

(a) All vector functions of the form (6) really are solutions to $\mathbf{x}' \;=\; A\,\mathbf{x}$.

This is the *superposition principle* for solutions of the system; it's true because the system is *linear*. The matrix notation makes it really easy to prove. We have

$$\begin{aligned}(c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n)' &= c_1\mathbf{x}_1' + \ldots + c_n\mathbf{x}_n' \\ &= c_1A\,\mathbf{x}_1 + \ldots + c_nA\,\mathbf{x}_n, && \text{since } \mathbf{x}_i' = A\,\mathbf{x}_i\ ; \\ &= A\,(c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n), && \text{by the distributive law (see LS.1).}\end{aligned}$$

(b) All solutions to the system are of the form (6).

This is harder to prove, and will be the main result of the next section.

2. The existence and uniqueness theorem for linear systems.

For simplicity, we stick with $n = 2$, but the results here are true for all n. There are two questions that need answering about the general linear system

$$\begin{aligned} x' &= a(t)x + b(t)y \\ y' &= c(t)x + d(t)y \end{aligned}\;; \qquad \text{in matrix form,} \quad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\;. \tag{2}$$

The first is from the previous section: to show that all solutions are of the form

$$\mathbf{x} = c_1\mathbf{x}_1 + x_2\mathbf{x}_2,$$

where the $\mathbf{x}_i$ form a fundamental set (i.e., neither is a constant multiple of the other). (The fact that we can write down *all* solutions to a linear system in this way is one of the main reasons why such systems are so important.)

An even more basic question for the system (2) is, how do we know that *has* two linearly independent solutions? For systems with a constant coefficient matrix A, we showed in the previous chapters how to solve them explicitly to get two independent solutions. But the general non-constant linear system (2) does not have solutions given by explicit formulas or procedures.

The answers to these questions are based on following theorem.

Theorem 5.2 Existence and uniqueness theorem for linear systems.

If the entries of the square matrix $A(t)$ are continuous on an open interval I containing t_0, then the initial value problem

$$\mathbf{x}' = A(t)\,\mathbf{x}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{7}$$

has one and only one solution $\mathbf{x}(t)$ on the interval I.

The proof is difficult, and we shall not attempt it. More important is to see how it is used. The three theorems following answer the questions posed, for the 2×2 system (2). They are true for $n > 2$ as well, and the proofs are analogous.

In the theorems, *we assume the entries of $A(t)$ are continuous on an open interval I*; then the conclusions are valid on the interval I. (For example, I could be the whole t-axis.)

Theorem 5.2A Linear independence theorem.

Let $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ be two solutions to (2) on the interval I, such that at some point t_0 in I, the vectors $\mathbf{x}_1(t_0)$ and $\mathbf{x}_2(t_0)$ are linearly independent. Then

a) the solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent on I, and

b) the vectors $\mathbf{x}_1(t_1)$ and $\mathbf{x}_2(t_1)$ are linearly independent at every point t_1 of I.

Proof. a) By contradiction. If they were dependent on I, one would be a constant multiple of the other, say $\mathbf{x}_2(t) = c_1\mathbf{x}_1(t)$; then $\mathbf{x}_2(t_0) = c_1\mathbf{x}_1(t_0)$, showing them dependent at t_0. □

b) By contradiction. If there were a point t_1 on I where they were dependent, say $\mathbf{x}_2(t_1) = c_1\mathbf{x}_1(t_1)$, then $\mathbf{x}_2(t)$ and $c_1\mathbf{x}_1(t)$ would be solutions to (2) which agreed at t_1, hence by the uniqueness statement in Theorem 5.2, $\mathbf{x}_2(t) = c_1\mathbf{x}_1(t)$ on all of I, showing them linearly dependent on I. □

Theorem 5.2B General solution theorem.

a) The system (2) has two linearly independent solutions.

b) If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are any two linearly independent solutions, then every solution $\mathbf{x}$ can be written in the form (8), for some choice of c_1 and c_2:

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2; \tag{8}$$

Proof. Choose a point $t = t_0$ in the interval I.

a) According to Theorem 5.2, there are two solutions $\mathbf{x}_1$, $\mathbf{x}_2$ to (3), satisfying respectively the initial conditions

$$\mathbf{x}_1(t_0) = \mathbf{i}\,, \qquad \mathbf{x}_2(t_0) = \mathbf{j}\ , \tag{9}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the usual unit vectors in the xy-plane. Since the two solutions are linearly independent when $t = t_0$, they are linearly independent on I, by Theorem 5.2A.

b) Let $\mathbf{u}(t)$ be a solution to (2) on I. Since $\mathbf{x}_1$ and $\mathbf{x}_2$ are independent at t_0 by Theorem 5.2, using the parallelogram law of addition we can find constants c_1' and c_2' such that

$$\mathbf{u}(t_0) = c_1'\mathbf{x}_1(t_0) + c_2'\mathbf{x}_2(t_0). \tag{10}$$

The vector equation (10) shows that the solutions $\mathbf{u}(t)$ and $c_1'\mathbf{x}_1(t) + c_2'\mathbf{x}_2(t)$ agree at t_0; therefore by the uniqueness statement in Theorem 5.2, they are equal on all of I, that is,

$$\mathbf{u}(t) = c_1'\mathbf{x}_1(t) + c_2'\mathbf{x}_2(t) \qquad \text{on } I.$$

3. The Wronskian

We saw in chapter LS.1 that a standard way of testing whether a set of n n-vectors are linearly independent is to see if the $n \times n$ determinant having them as its rows or columns is non-zero. This is also an important method when the n-vectors are solutions to a system; the determinant is given a special name. (Again, we will assume $n = 2$, but the definitions and results generalize to any n.)

Definition 5.3 *Let $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ be two 2-vector functions. We define their* **Wronskian** *to be the determinant*

$$W(\mathbf{x}_1, \mathbf{x}_2)(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix} \tag{11}$$

whose columns are the two vector functions.

The independence of the two vector functions should be connected with their Wronskian not being zero. At least for points, the relationship is clear; using the result mentioned above, we can say

$$W(\mathbf{x}_1, \mathbf{x}_2)(t_0) = \begin{vmatrix} x_1(t_0) & x_2(t_0) \\ y_1(t_0) & y_2(t_0) \end{vmatrix} = 0 \quad \Leftrightarrow \quad \mathbf{x}_1(t_0) \text{ and } \mathbf{x}_2(t_0) \textit{ are dependent.} \tag{12}$$

However for vector functions, the relationship is clear-cut *only when* $\mathbf{x}_1$ *and* x_2 *are solutions to a well-behaved ODE sysem (2).* The theorem is:

Theorem 5.3 Wronskian vanishing theorem.

On an interval I where the entries of $A(t)$ are continuous, let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two solutions to (2), and $W(t)$ their Wronskian (11). Then either

a) $W(t) \equiv 0$ on I, and $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly dependent on I, or

b) $W(t)$ is never 0 on I, and $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent on I.

Proof. Using (12), there are just two possibilities.

a) $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly dependent on I; say $\mathbf{x}_2 = c_1\mathbf{x}_1$. In this case they are dependent at each point of I, and $W(t) \equiv 0$ on I, by (12);

b) $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent on I, in which case by Theorem 5.2A they are linearly independent at each point of I, and so $W(t)$ is never zero on I, by (12). $\square$

Exercises: Section 4E

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