LS.4   Decoupling Systems

1.   Changing variables.

A common way of handling mathematical models of scientific or engineering problems is to look for a change of coordinates or a change of variables which simplifies the problem. We handled some types of first-order ODE's — the Bernouilli equation and the homogeneous equation, for instance — by making a change of dependent variable which converted them into equations we already knew how to solve. Another example would be the use of polar or spherical coordinates when a problem has a center of symmetry.

An example from physics is the description of the acceleration of a particle moving in the plane: to get insight into the acceleration vector, a new coordinate system is introduced whose basis vectors are $\mathbf{t}$ and $\mathbf{n}$ (the unit tangent and normal to the motion), with the result that $\mathbf{F} = m\mathbf{a}$ becomes simpler to handle.

We are going to do something like that here. Starting with a homogeneous linear system with constant coefficients, we want to make a linear change of coordinates which simplifies the system. We will work with $n = 2$, though what we say will be true for $n > 2$ also.

How would a simple system look? The simplest system is one with a diagonal matrix: written first in matrix form and then in equation form, it is

$$(1) \qquad \begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \text{or} \qquad \begin{matrix} u' = \lambda_1 u \\ v' = \lambda_2 v \end{matrix} \ .$$

As you can see, if the coefficient matrix has only diagonal entries, the resulting "system" really consists of a set of first-order ODE's, side-by-side as it were, each involving only its own variable. Such a system is said to be **decoupled** since the variables do not interact with each other; each variable can be solved for independently, without knowing anything about the others. Thus, solving the system on the right of (1) gives

$$(2) \qquad \begin{matrix} u = c_1 e^{\lambda_1 t} \\ v = c_2 e^{\lambda_2 t} \end{matrix} \ , \qquad \text{or} \qquad \mathbf{u} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\lambda_2 t}.$$

So we start with a 2×2 homogeneous system with constant coefficients,

$$(3) \qquad \qquad \mathbf{x}' = A\mathbf{x} \ ,$$

and we want to introduce new dependent variables u and v, related to x and y by a linear change of coordinates, i.e., one of the form (we write it three ways):

$$(4) \qquad \mathbf{u} = D\mathbf{x} \ , \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \ , \qquad \begin{matrix} u = ax + by \\ v = cx + dy \end{matrix} \ .$$

We call D the **decoupling matrix**. After the change of variables, we want the system to be decoupled, i.e., to look like the system (1). What should we choose as D?

The matrix D will define the new variables u and v in terms of the old ones x and y. But in order to substitute into the system (3), it is really the inverse to D that we need; we shall denote it by E :

(5) $$\mathbf{u} \; = \; D\,\mathbf{x}, \qquad \mathbf{x} \; = \; E\,\mathbf{u}\,, \qquad E \; = \; D^{-1}\,.$$

In the decoupling, we first produce E ; then D is calculated as its inverse. We need both matrices: D to define the new variables, E to do the substitutions.

We are now going to assume that the ODE system $\mathbf{x}' = A\mathbf{x}$ has *two real and distinct eigenvalues*; with their associated eigenvectors, they are denoted as usual in these notes by

(6) $$\lambda_1, \quad \vec{\alpha}_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \; ; \qquad \lambda_2, \quad \vec{\alpha}_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\,.$$

The idea now is the following. Since these eigenvectors are somehow "special" to the system, let us choose the new coordinates so that the eigenvectors become the unit vectors $\mathbf{i}$ and $\mathbf{j}$ in the uv-system. To do this, we make the eigenvectors the two columns of the matrix E ; that is, we make the change of coordinates

(7) $$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}\,, \qquad E = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\,.$$

With this choice for the matrix E,

$$\mathbf{i} \; = \; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{j} \; = \; \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

in the uv-system correspond in the xy-system respectively to the first and second columns of E, as you can see from (7).

We now have to show that this change to the uv-system decouples the ODE system $\mathbf{x}' = A\mathbf{x}$. This rests on the following very important equation connecting a matrix A, one of its eigenvalues λ, and a corresponding eigenvector $\vec{\alpha}$:

(8) $$A\,\vec{\alpha} \; = \; \lambda\,\vec{\alpha}\,,$$

which follows immediately from the equation used to calculate the eigenvector:

$$(A - \lambda I)\,\vec{\alpha} \; = \; 0 \qquad \Rightarrow \qquad A\,\vec{\alpha} \; = \; (\lambda I)\,\vec{\alpha} \; = \; \lambda(I\,\vec{\alpha}) \; = \; \lambda\,\vec{\alpha}\,.$$

> The equation (8) is often used as the definition of eigenvector and eigenvalue: an *eigenvector of A* is a vector which changes by some scalar factor λ when multiplied by A; the factor λ is the *eigenvalue* associated with the vector.

As it stands, (8) deals with only one eigenvector at a time. We recast it into the standard form in which it deals with both eigenvectors simultaneously. Namely, (8) says that

$$A \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \; = \; \lambda_1 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}\,, \qquad A \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \; = \; \lambda_2 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\,.$$

These two equations can be combined into the single matrix equation

(9) $$A \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \; = \; \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\,, \qquad \text{or} \qquad A\,E \; = \; E \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\,,$$

as is easily checked. Note that the diagonal matrix of λ's must be placed on the right in order to multiply the *columns* by the λ's; if we had placed it on the left, it would have multiplied the *rows* by the λ's, which is not what we wanted.

From this point on, the rest is easy. We want to show that the change of variables $\mathbf{x} = E\,\mathbf{u}$ decouples the system $\mathbf{x}' = A\,\mathbf{x}$, where E is defined by (7). We have, substituting $\mathbf{x} = E\,\mathbf{u}$ into the system, the successive equations

$$\mathbf{x}' = A\,\mathbf{x}$$
$$E\,\mathbf{u}' = A\,E\,\mathbf{u}$$
$$E\,\mathbf{u}' = E\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\mathbf{u}, \qquad \text{by (9);}$$

multiplying both sides on the left by $D = E^{-1}$ then shows the system is decoupled:

$$\mathbf{u}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\mathbf{u} \quad .$$

Definition. *For a matrix A with two real and distinct eigenvalues, the matrix E in (7) whose columns are the eigenvectors of A is called an* **eigenvector matrix** *for A , and the matrix $D = E^{-1}$ is called the* **decoupling matrix** *for the system $\mathbf{x}' = A\,\mathbf{x}$; the new variables u, v in (7) are called the* **canonical variables**.

One can alter the matrices by switching the columns, or multiplying a column by a non-zero scalar, with a corresponding alteration in the new variables; apart from that, they are unique.

Example 1. For the system

$$x' = x - y$$
$$y' = 2x + 4y$$

make a linear change of coordinates which decouples the system; verify by direct substitution that the system becomes decoupled.

Solution. In matrix form the system is $\quad \mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$.

We calculate first E, as defined by (7); for this we need the eigenvectors. The characteristic polynomial of A is

$$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) ;$$

the eigenvalues and corresponding eigenvectors are, by the usual calculation,

$$\lambda_1 = 2, \quad \vec{\alpha}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} ; \qquad \lambda_2 = 3, \quad \vec{\alpha}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} .$$

The matrix E has the eigenvectors as its columns; then $D = E^{-1}$. We get (cf. LS.1, (2) to calculate the inverse matrix to E)

$$E = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} , \qquad\qquad D = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} .$$

By (4), the new variables are defined by

$$\mathbf{u} = D\,\mathbf{x} , \qquad\qquad \begin{aligned} u &= 2x + y \\ v &= -x - y . \end{aligned}$$

To substitute these into the system and check they they decouple we use

$$\mathbf{x} \; = \; E\,\mathbf{u}\;, \qquad \begin{aligned} x &= \; u+v \\ y &= \; -u-2v \end{aligned}\;.$$

Substituting these into the original system (on the left below) gives us the pair of equations on the right:

$$\begin{aligned} x' &= \; x-y \\ y' &= \; 2x+4y \end{aligned} \qquad \begin{aligned} u'+v' &= \; 2u+3v \\ -u'-2v' &= \; -2u-6v \end{aligned} \quad ;$$

adding the equations eliminates u; multiplying the top equation by 2 and adding eliminates v, giving the system

$$\begin{aligned} u' &= \; 2y \\ v' &= \; 3v \end{aligned}$$

which shows that in the new coordinates the system is decoupled.

The work up to this point assumes that $n = 2$ and the eigenvalues are real and distinct. What if this is not so?

If the eigenvalues are complex, the corresponding eigenvectors will also be complex, i.e., have complex components. All of the above remains formally true, provided we allow all the matrices to have complex entries. This means the new variables u and v will be expressed in terms of x and y using complex coefficients, and the decoupled system will have complex coefficients.

In some branches of science and engineering, this is all perfectly acceptable, and one gets in this way a complex decoupling. If one insists on using real variables only, a decoupling is not possible.

If there is only one (repeated) eigenvalue, there are two cases, as discussed in LS.3 . In the complete case, there are two independent eigenvalues, but as pointed out there (Theorem 3.2), the system will be be automatically decoupled, i.e. A will be a diagonal matrix. In the incomplete case, there is only one eigenvector, and decoupling is impossible (since in the decoupled system, both $\mathbf{i}$ and $\mathbf{j}$ would be eigenvectors).

For $n \geq 3$, real decoupling requires us to find n linearly independent real eigenvectors, to form the columns of the nonsingular matrix E. This is possible if

a) all the eigenvalues are real and distinct, or

b) all the eigenvalues are real, and each repeated eigenvalue is complete.

Repeating the end of LS.3, we note again the important theorem in linear algebra which guarantees decoupling is possible:

Theorem. *If the matrix A is real and symmetric, i.e., $A^T = A$, all its eigenvalues will be real and complete, so that the system $\mathbf{x}' = A\mathbf{x}$ can always be decoupled.*

Exercises: Section 4E

M.I.T. 18.03 Ordinary Differential Equations
18.03 Notes and Exercises