LS.3 Complex and Repeated Eigenvalues

1. Complex eigenvalues.

In the previous chapter, we obtained the solutions to a homogeneous linear system with constant coefficients

\[ A \mathbf{x} = 0 \]

under the assumption that the roots of its characteristic equation \(|A - \lambda I| = 0\) — i.e., the eigenvalues of \(A\) — were real and distinct.

In this section we consider what to do if there are complex eigenvalues. Since the characteristic equation has real coefficients, its complex roots must occur in conjugate pairs:

\[ \lambda = a + bi, \quad \bar{\lambda} = a - bi. \]

Let’s start with the eigenvalue \(a + bi\). According to the solution method described in Chapter LS.2, the next step would be to find the corresponding eigenvector \(\vec{\alpha}\), by solving the equations (LS.2, (27))

\[
\begin{align*}
(a - \lambda)a_1 + ba_2 &= 0 \\
ca_1 + (d - \lambda)a_2 &= 0
\end{align*}
\]

for its components \(a_1\) and \(a_2\). Since \(\lambda\) is complex, the \(a_i\) will also be complex, and therefore the eigenvector \(\vec{\alpha}\) corresponding to \(\lambda\) will have complex components.

Putting together the eigenvalue and eigenvector gives us formally the complex solution

\[ \mathbf{x} = \vec{\alpha} e^{(a+bi)t}. \]

Naturally, we want real solutions to the system, since it was real to start with. To get them, the following theorem tells us to just take the real and imaginary parts of (1).

**Theorem 3.1** Given a system \(\mathbf{x}' = A\mathbf{x}\), where \(A\) is a real matrix. If \(\mathbf{x} = x_1 + i x_2\) is a complex solution, then its real and imaginary parts \(x_1, x_2\) are also solutions to the system.

**Proof.** Since \(x_1 + i x_2\) is a solution, we have

\[ (x + i x_2)' = A(x + i x_2); \]

equating real and imaginary parts of this equation,

\[ x_1' = A x_1, \quad x_2' = A x_2, \]

which shows that the real vectors \(x_1\) and \(x_2\) are solutions to \(\mathbf{x}' = A\mathbf{x}.\)

**Example 1.** Find the corresponding two real solutions to \(\mathbf{x}' = A\mathbf{x}\) if a complex eigenvalue and corresponding eigenvector are

\[ \lambda = -1 + 2i, \quad \vec{\alpha} = \begin{pmatrix} i \\ 2 - 2i \end{pmatrix}. \]

**Solution.** First write \(\vec{\alpha}\) in terms of its real and imaginary parts:

\[ \vec{\alpha} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + i \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \]
The corresponding complex solution $x = \vec{\alpha} e^{\lambda t}$ to the system can then be written
\[ x = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + i \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} (\cos 2t + i \sin 2t) , \]
so that we get respectively for the real and imaginary parts of $x$
\[ x_1 = e^{-t} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos 2t - i \begin{pmatrix} 1 \\ -2 \end{pmatrix} \sin 2t = e^{-t} \begin{pmatrix} -\sin 2t \\ 2 \cos 2t + 2 \sin 2t \end{pmatrix} , \]
\[ x_2 = e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cos 2t - i \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sin 2t = e^{-t} \begin{pmatrix} -\cos 2t \\ -2 \cos 2t + 2 \sin 2t \end{pmatrix} ; \]
these are the two real solutions to the system.

In general, if the complex eigenvalue is $a + bi$, to get the real solutions to the system, we write the corresponding complex eigenvector $\vec{\alpha}$ in terms of its real and imaginary part:
\[ \vec{\alpha} = \vec{\alpha}_1 + i \vec{\alpha}_2, \quad \vec{\alpha}_i \text{ real vectors;} \]
(study carefully in the above example how this is done in practice). Then we substitute into (1) and calculate as in the example:
\[ x = (\vec{\alpha}_1 + i \vec{\alpha}_2) e^{at} (\cos bt + i \sin bt) , \]
so that the real and imaginary parts of $x$ give respectively the two real solutions
\[ x_1 = e^{at} (\vec{\alpha}_1 \cos bt - \vec{\alpha}_2 \sin bt) , \]
\[ x_2 = e^{at} (\vec{\alpha}_1 \sin bt + \vec{\alpha}_2 \cos bt) . \]
These solutions are linearly independent if $n = 2$. If $n > 2$, that portion of the general solution corresponding to the eigenvalues $a \pm bi$ will be
\[ c_1 x_1 + c_2 x_2 . \]

Note that, as for second-order ODE’s, the complex conjugate eigenvalue $a - bi$ gives up to sign the same two solutions $x_1$ and $x_2$.

The expression (3) was not written down for you to memorize, learn, or even use; the point was just for you to get some practice in seeing how a calculation like that in Example 1 looks when written out in general. To actually solve ODE systems having complex eigenvalues, imitate the procedure in Example 1.

Stop at this point, and practice on an example (try Example 3, p. 377).

2. Repeated eigenvalues. Again we start with the real $n \times n$ system
\[ x' = Ax . \]
We say an eigenvalue $\lambda_1$ of $A$ is repeated if it is a multiple root of the characteristic equation of $A$—in other words, the characteristic polynomial $|A - \lambda I|$ has $(\lambda - \lambda_1)^2$ as a factor. Let’s suppose that $\lambda_1$ is a double root; then we need to find two linearly independent solutions to the system (4) corresponding to $\lambda_1$.

One solution we can get: we find in the usual way an eigenvector $\vec{\alpha}_1$ corresponding to $\lambda_1$ by solving the system
\[ (A - \lambda_1 I) \mathbf{a} = 0 . \]
This gives the solution $x_1 = \vec{\alpha}_1 e^{\lambda_1 t}$ to the system (4). Our problem is to find a second solution. To do this we have to distinguish two cases. The first one is easy.
A. The complete case.

Still assuming $\lambda_1$ is a real double root of the characteristic equation of $A$, we say $\lambda_1$ is a complete eigenvalue if there are two linearly independent eigenvectors $\vec{\alpha}_1$ and $\vec{\alpha}_2$ corresponding to $\lambda_1$; i.e., if these two vectors are two linearly independent solutions to the system (5). Using them we get two independent solutions to (4), namely

$$x_1 = \vec{\alpha}_1 e^{\lambda_1 t}, \quad \text{and} \quad x_2 = \vec{\alpha}_2 e^{\lambda_1 t}.$$

Naturally we would like to see an example of this for $n = 2$, but the following theorem explains why there aren’t any good examples: if the matrix $A$ has a repeated eigenvalue, it is so simple that no one would solve the system (4) by fussing with eigenvectors!

**Theorem 3.2** Let $A$ be a $2 \times 2$ matrix and $\lambda_1$ an eigenvalue. Then

$\lambda_1$ is repeated and complete $\iff A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$

**Proof.** Let $\vec{\alpha}_1$ and $\vec{\alpha}_2$ be two independent solutions to (5). Then any 2-vector $\vec{\alpha}$ is a solution to (5), for by using the parallelogram law of addition, $\vec{\alpha}$ can be written in the form

$$\vec{\alpha} = c_1 \vec{\alpha}_1 + c_2 \vec{\alpha}_2,$$

and this shows it is a solution to (5), since $\vec{\alpha}_1$ and $\vec{\alpha}_2$ are:

$$(A - \lambda_1 I)\vec{\alpha} = c_1 (A - \lambda_1 I)\vec{\alpha}_1 + c_2 (A - \lambda_1 I)\vec{\alpha}_2 = 0 + 0.$$

In particular, the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ satisfy (5); letting $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\begin{pmatrix} a - \lambda_1 & b \\ c & d - \lambda_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a = \lambda_1 \\ c = 0 \end{cases};$$

$$\begin{pmatrix} a - \lambda_1 & b \\ c & d - \lambda_1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} b = 0 \\ d = \lambda_1 \end{cases}.$$

This proves the theorem in the direction $\Rightarrow$; in the other direction, one sees immediately that the characteristic polynomial is $(\lambda - \lambda_1)^2$, so that $\lambda_1$ is a repeated eigenvalue; it is complete since the matrix $A - \lambda_1 I$ has 0 for all its entries, and therefore every 2-vector $\vec{\alpha}$ is a solution to (5).

For $n = 3$ the situation is more interesting. Still assuming $\lambda_1$ is a double root of the characteristic equation, it will be a complete eigenvalue when the system (5) has two independent solutions; this will happen when the system (5) has essentially only one equation: the other two equations are constant multiples of it (or identically 0). You can then find two independent solutions to the system just by inspection.

**Example 2.** If the system (5) turns out to be three equations, each of which is a constant multiple of say

$$2a_1 - a_2 + a_3 = 0,$$

we can give $a_1$ and $a_2$ arbitrary values, and then $a_3$ will be determined by the above equation. Hence two independent solutions (eigenvectors) would be the column 3-vectors

$$(1, 0, 2)^T \quad \text{and} \quad (0, 1, 1)^T.$$

In general, if an eigenvalue $\lambda_1$ of $A$ is $k$-tuply repeated, meaning the polynomial $A - \lambda I$ has the power $(\lambda - \lambda_1)^k$ as a factor, but no higher power, the eigenvalue is called **complete** if it
has $k$ independent associated eigenvectors, i.e., if the system (5) has $k$ linearly independent solutions. These then produce $k$ solutions to the ODE system (4).

There is an important theorem in linear algebra (it usually comes at the very end of a linear algebra course) which guarantees that all the eigenvalues of $A$ will be complete, regardless of what their multiplicity is:

**Theorem 3.3** If the real square matrix $A$ is symmetric, meaning $A^T = A$, then all its eigenvalues are real and complete.

**B. The defective case.**

If the eigenvalue $\lambda$ is a double root of the characteristic equation, but the system (5) has only one non-zero solution $\vec{\alpha}_1$ (up to constant multiples), then the eigenvalue is said to be **incomplete** or **defective**, and no second eigenvector exists. In this case, the second solution to the system (4) has a different form. It is

$$x = (\vec{\beta} + \vec{\alpha}_1 t)e^{\lambda_1 t},$$

where $\vec{\beta}$ is an unknown vector which must be found. This may be done by substituting (7) into the system (4), and using the fact that $\vec{\alpha}_1$ is an eigenvector, i.e., a solution to (5) when $\lambda = \lambda_1$. When this is done, we find that $\vec{\beta}$ must be a solution to the system

$$\left( A - \lambda_1 I \right) \vec{\beta} = \vec{\alpha}_1.$$

This is an inhomogeneous system of equations for determining $\vec{\beta}$. It is guaranteed to have a solution, provided that the eigenvalue $\lambda_1$ really is defective.

Notice that (8) doesn’t look very solvable, because the matrix of coefficients has determinant zero! So you won’t solve it by finding the inverse matrix or by using Cramer’s rule. It has to be solved by elimination.

Some people do not bother with (7) or (8); when they encounter the defective case (at least when $n = 2$), they give up on eigenvalues, and simply solve the original system (4) by elimination.

**Example.** Try Example 2 (section 5.6) in your book; do it by using (7) and (8) above to find $\vec{\beta}$, then check your answer by instead using elimination to solve the ODE system.

**Proof of (7) for $n = 2$.** Let $A$ be a $2 \times 2$ matrix.

Since $\lambda_1$ is to be a double root of the characteristic equation, that equation must be

$$(\lambda - \lambda_1)^2 = 0,$$

i.e.,

$$\lambda^2 - 2\lambda_1 \lambda + \lambda_1^2 = 0.$$

From the known form of the characteristic equation (LS.2, (25)), we see that

$$\text{trace } A = 2\lambda_1, \quad \det A = \lambda_1^2.$$

A convenient way to write two numbers whose sum is $2\lambda_1$ is: $\lambda_1 \pm a$; doing this, we see that our matrix $A$ takes the form

$$A = \begin{pmatrix} \lambda_1 + a & b \\ c & \lambda_1 - a \end{pmatrix}, \quad \text{where } bc = -a^2, \quad \text{(since } \det A = \lambda_1^2).$$
Now calculate the eigenvectors of such a matrix $A$. Note that $b$ and $c$ are not both zero, for if they were, $a = 0$ by (9), and the eigenvalue would be complete. If say $b \neq 0$, we may choose as the eigenvector

$$\tilde{\alpha}_1 = \begin{pmatrix} b \\ -a \end{pmatrix},$$

and then by (8), we get

$$\beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

**Exercises:** Section 4D
M.I.T. 18.03 Ordinary Differential Equations
18.03 Notes and Exercises