Concourse 18.03 – Lecture #8

Today we continue with some of the details of 2nd order (and higher) linear ODE's. We will focus primarily on the constant coefficient case in which we can easily construct exponential solutions from the roots of the characteristic equation and then combine these solutions via superposition. Linear algebra idea regarding spanning sets and linear independence will be introduced as needed, and we'll define the Wronskian as a tool for checking independence of solutions. We'll apply these methods to the study of mass-spring-dashpot systems.

Higher order linear ordinary differential equations with constant coefficients

In general, an *n*th order linear ordinary differential equation is a differential equation of the form $\frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\frac{dx}{dt} + p_0(t)x(t) = q(t)$, where $p_{n-1}(t),\ldots,p_1(t),p_0(t),q(t)$ are functions of the independent variable *t*. We solve this by (1) finding an expression for all homogeneous solutions $x_h(t)$, (2) using some productive method to find one particular solution $x_p(t)$ to the inhomogeneous equation, and then (3) adding these to get the general solution $x(t) = x_h(t) + x_p(t)$. If we are solving an initial value problem, we would then use the initial conditions to determine any unknown constants in the expression for $x(t)$.

One case of special interest is the case where all of the coefficient functions $p_i(t) = a_i$ are constant. In this case the differential equation simplifies to $\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1\frac{dx}{dt} + a_0 x(t) = q(t)$. If we write $D = \frac{d}{dt}$, $D^2 = D \circ D = \frac{d^2}{dt^2}$, etc. and $I = Identity$, we can express this ODE as $\left[D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 I\right]x(t) = q(t)$. Note that this **linear operator** $T = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 I$ has a very polynomial-like quality. It has a corresponding **characteristic polynomial** $p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ that permits us to formally express $T = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 I = p(D)$. We will often write such an ODE in the form $\left[p(D)\right]x(t) = q(t)$.

If we seek exponential solutions of the form e^{rt} for the homogeneous equation $\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1\frac{dx}{dt} + a_0 x(t) = 0$, we calculate $\frac{dx}{dt} = re^{rt}$, $\frac{d^2x}{dt^2} = r^2 e^{rt}$, …, $\frac{d^n x}{dt^n} = r^n e^{rt}$, and substitution gives $r^n e^{rt} + a_{n-1}r^{n-1}e^{rt} + \cdots + a_2 r^2 e^{rt} + a_1 r e^{rt} + a_0 e^{rt} = (r^n + a_{n-1}r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0)e^{rt} = p(r)e^{rt} = 0$. This yields a solution only when the characteristic polynomial $p(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0$.

So for any root r_i of the characteristic polynomial, $e^{r_i t}$ will be a homogeneous solution. The Fundamental Theorem of Algebra guarantees (in principle) that we can factor $p(r)$ into a product of linear factors and irreducible quadratic factors. As long as there are no repeated roots, and since we can use the quadratic formula to produce a complex conjugate pair of roots for each irreducible quadratic factor, we will be able to produce *n* distinct roots and a corresponding set of exponential solutions $\left\{e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}\right\}$. In the case of repeated roots, this will yield fewer solutions of this form.

By linearity, any function of the form $x_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \ldots + c_n e^{r_n t}$ will solve the homogeneous equation.

Question: Does this yield <u>all</u> solutions?

A second order example should explain why the answer is <u>YES</u>. Suppose we wish to solve the ODE $\ddot{x} + 3\dot{x} + 2x = 0$. Any exponential solution e^{rt} would give $p(r) = r^2 + 3r + 2 = (r+2)(r+1) = 0$. Its characteristic roots are $r_1 = -2$ and $r_2 = -1$, and these yield solutions e^{-2t} and e^{-t}. Why are ALL homogeneous solutions of the form $x(t) = c_1 e^{-2t} + c_2 e^{-t}$?

If we write the differential equation in terms of linear differential operators, we might write this as $\left[D+2I\right]\circ\left[D+I\right]x(t)=0$, i.e. as a composition of two 1st order linear differential operators. If we let $\left[D+I\right]x(t)=y(t)$, this gives two 1st order equations: $\frac{dx}{dt}+x=y(t)$ and $\frac{dy}{dt}+2y=0$. The latter equation is easily solved to give all solutions $y(t)=c_1e^{-2t}$ where c_1 is a constant. We then substitute this into the former equation to get $\frac{dx}{dt}+x=c_1e^{-2t}$. This is an inhomogeneous equation with integrating factor e^t. Multiplication by this gives $e^t\frac{dx}{dt}+e^tx=\frac{d}{dt}(e^tx)=c_1e^{-t}$, so $e^tx(t)=-c_1e^{-t}+c_2$. Finally, multiplying both sides by e^{-t} gives $x(t)=-c_1e^{-2t}+c_2e^{-t}$. Except for the sign switch on the first arbitrary constant, this demonstrates that ***all*** homogeneous solutions are of the form $x(t)=c_1e^{-2t}+c_2e^{-t}$ for some choices of the constants c_1 and c_2, i.e. all linear combinations of the two basic exponential solutions that we found.

It should be clear that this approach can be generalized to the *n*th order case as long as the characteristic polynomial can be factored into distinct linear factors. (We write the differential equation as a composition of *n* 1st order linear operators and iterate the above process.) This even works in the case of complex roots as long as they are not repeated. The more difficult case is when there are repeated roots of the characteristic polynomial, but, as we'll soon see, this case also yields a relatively simple solution.

In Linear Algebra terms, we say that $\left\{e^{r_1t},e^{r_2t},\dots,e^{r_nt}\right\}$ **span all solutions** in the above case. It is a valid question to ask whether all of these solutions are necessary, i.e. if we could span all solutions with a subset of these exponential solutions. In Linear Algebra terms, we would ask: **Are these solutions are linearly independent**? In other words, is it possible to express any of these solutions as a linear combination of the other solutions?

Definition: A set of functions $\{f_1,f_2,\dots,f_n\}$ is called **linearly independent** if the equation $c_1f_1(t)+c_2f_2(t)+\ldots+c_nf_n(t)=0$ (for all *t*) implies that $c_1=c_2=\ldots=c_n=0$.

When seeking solutions to an *n*th order linear differential equation of the form $\left[p(D)\right]x(t)=q(t)$, we actually **want more than this**. We want to guarantee a unique solution to any well-posed initial value problem with initial conditions given for the function and its derivatives up to order $(n-1)$, i.e. $x(t_0)=x_0$, $\dot{x}(t_0)=\dot{x}_0$, … $x^{(n-1)}(t_0)=x_0^{(n-1)}$. If $x_p(t)$ is one particular solution and if we can express all homogeneous solutions as $x_h(t)=c_1f_1(t)+c_2f_2(t)+\cdots+c_nf_n(t)$,then we would have the general solution $x(t)=x_h(t)+x_p(t)=c_1f_1(t)+c_2f_2(t)+\cdots+c_nf_n(t)+x_p(t)$ and we would then also want that:

$$\left\{\begin{matrix} x_h(t_0)+x_p(t_0)=x(t_0) \\ \dot{x}_h(t_0)+\dot{x}_p(t_0)=\dot{x}(t_0) \\ \vdots \\ x_h^{(n-1)}(t_0)+x_p^{(n-1)}(t_0)=x^{(n-1)}(t_0)\end{matrix}\right\}\Rightarrow\left\{\begin{matrix} c_1f_1(t_0)+c_2f_2(t_0)+\cdots+c_nf_n(t_0)+x_p(t_0)=x(t_0) \\ c_1f_1'(t_0)+c_2f_2'(t_0)+\cdots+c_nf_n'(t_0)+\dot{x}_p(t_0)=\dot{x}(t_0) \\ \vdots \\ c_1f_1^{(n-1)}(t_0)+c_2f_2^{(n-1)}(t_0)+\cdots+c_nf_n^{(n-1)}(t_0)+x_p^{(n-1)}(t_0)=x^{(n-1)}(t_0)\end{matrix}\right\}$$

To guarantee a unique solution to the initial value problem, we would have to produce unique values for $\{c_1,c_2,\dots,c_n\}$. We can rewrite the above system of linear equations in the form:

$$\left\{\begin{matrix} f_1(t_0)c_1+f_2(t_0)c_2+\cdots+f_n(t_0)c_n=x(t_0)-x_p(t_0) \\ f_1'(t_0)c_1+f_2'(t_0)c_2+\cdots+f_n'(t_0)c_n=\dot{x}(t_0)-\dot{x}_p(t_0) \\ \vdots \\ f_1^{(n-1)}(t_0)c_1+f_2^{(n-1)}(t_0)c_2+\cdots+f_n^{(n-1)}(t_0)c_n=x^{(n-1)}(t_0)-x_p^{(n-1)}(t_0)\end{matrix}\right\}$$

In terms of matrices, we can express these as:

$$\begin{bmatrix} f_1(t_0) & f_2(t_0) & \cdots & f_n(t_0) \\ f_1'(t_0) & f_2'(t_0) & \cdots & f_n'(t_0) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(t_0) & f_2^{(n-1)}(t_0) & \cdots & f_n^{(n-1)}(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} x(t_0)-x_p(t_0) \\ \dot{x}(t_0)-\dot{x}_p(t_0) \\ \vdots \\ x^{(n-1)}(t_0)-x_p^{(n-1)}(t_0) \end{bmatrix} \Rightarrow \text{ unique } \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Two fundamental results in linear algebra say that <u>this will only be the case when the above matrix is invertible</u>, and <u>this will only be the case when its determinant is never equal to 0</u>.

Definition: $\det \begin{bmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{bmatrix} = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{vmatrix} = W(t)$

is called the **Wronskian determinant**.

Corollary: If the Wronskian determinant is never 0, the given ODE will yield unique solutions in the form $x(t) = x_h(t) + x_p(t) = c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t) + x_p(t)$ for any given initial conditions given for the function and its derivatives up to order $(n-1)$.

Though not routinely used to ensure a linearly independent set of solutions, (there are arguments with less tedious calculations that can be made), the Wronskian is one tool for ensuring that a set of homogeneous solutions to a linear ODE is valid for ***uniquely*** expressing ***all*** solutions to a given initial value problem.

Example: Solve the initial value problem $\ddot{x} + 5\dot{x} + 4x = 3\sin 2t$ with initial conditions $x(0) = 3$, $\dot{x}(0) = 2$.

Solution: We first solve the homogeneous equation $\ddot{x} + 5\dot{x} + 4x = 0$. Its characteristic polynomial is $p(r) = r^2 + 5r + 4 = (r+4)(r+1)$ and this yields two distinct roots $r = -4$ and $r = -1$. The corresponding exponential solutions are e^{-4t} and e^{-t}. We can check that these are, in fact, linearly independent by calculating the Wronskian determinant: $\begin{vmatrix} e^{-4t} & e^{-t} \\ -4e^{-4t} & -e^{-t} \end{vmatrix} = -e^{-5t} + 4e^{-5t} = 3e^{-5t} \neq 0$. From our previous arguments, we know that all homogeneous solutions are of the form $x_h(t) = c_1 e^{-4t} + c_2 e^{-t}$.

Next, we seek a particular solution. There are at least two good ways to do this. We could do "complex replacement" and simultaneously solve $\ddot{x} + 5\dot{x} + 4x = 3\cos 2t$ and $\ddot{y} + 5\dot{y} + 4y = 3\sin 2t$ by solving the inhomogeneous equation $\ddot{z} + 5\dot{z} + 4z = 3e^{2it}$ and then taking the "imaginary" part. It is perhaps easier to solve using undetermined coefficients.

If we let $x = a\cos 2t + b\sin 2t$, we get $\begin{Bmatrix} x = a\cos 2t + b\sin 2t \\ \dot{x} = 2b\cos 2t - 2a\sin 2t \\ \ddot{x} = -4a\cos 2t - 4b\sin 2t \end{Bmatrix} \Rightarrow \ddot{x} + 5\dot{x} + 4x = (10b)\cos 2t + (-10a)\sin 2t$

We must therefore have $10b = 0$ and $-10a = 3$, so $a = -\frac{3}{10}$ and $b = 0$. So $x_p(t) = -\frac{3}{10}\cos 2t$.

The general solution is therefore $x(t) = c_1 e^{-4t} + c_2 e^{-t} - \frac{3}{10}\cos 2t$, and we have $\dot{x}(t) = -4c_1 e^{-4t} - c_2 e^{-t} + \frac{3}{5}\cos 2t$.

If we substitute the initial conditions $x(0) = 3$, $\dot{x}(0) = 2$, we have:

$$\begin{Bmatrix} x(0) = c_1 + c_2 - \frac{3}{10} = 3 \\ \dot{x}(0) = -4c_1 - c_2 = 2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} c_1 + c_2 = \frac{33}{10} \\ -4c_1 - c_2 = 2 \end{Bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{33}{10} \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{53}{30} \\ \frac{76}{15} \end{bmatrix}.$$

We certainly don't have to use matrices to solve these two equations, but it's worth noting that the nonvanishing of the Wronskian determinant is precisely why there is a unique solution for these constants. The unique solution to this initial value problem is therefore $\boxed{x(t) = -\frac{53}{30}e^{-4t} + \frac{76}{15}e^{-t} - \frac{3}{10}\cos 2t}$.

Note: In this example, the exponential terms are transients (they decay quickly) and the "steady state" solution is the particular solution that we calculated.

Mass-Spring-Dashpot systems

Of particular interest to us (for a variety of reasons) are mass-spring-dashpot systems in which a spring is governed by Hooke's Law but also subject to friction that is proportional to the velocity. [A picture was drawn in class illustrating a spring with an attached mass and the friction supplied by a piston (dashpot).] The simplest case is where this system is confined with the spring attached to one fixed wall, the dashpot on the other side of the mass attached to another fixed wall, and the mass moving relative to its equilibrium position. In this case, we would express the force acting on the mass as $F = -kx - cv$ where $v = \dot{x}$ and $F = ma = m\ddot{x}$. This gives the system $m\ddot{x} + c\dot{x} + kx = 0$ or $\boxed{\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0}$.

We could also imagine a system that is "driven" by moving either the fixed end of the spring or by moving the fixed end of the dashpot. If we incorporate this additional acceleration, the resulting system would be governed by an inhomogeneous ODE of the form $\boxed{\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = q(t)}$.

Note: We get similar equations in the case of an electric circuit with inductance (L), resistance (R), and capacitance (C), i.e. and ***LRC* circuit**.

Spring only case

The simplest case is a pure spring with no friction and no external driving force. In this case, the differential equation governing the motion would be simply $\boxed{\ddot{x} + \frac{k}{m}x = 0}$. In anticipation of what will follow, it's useful to let $\omega^2 = \frac{k}{m}$ or $\omega = \sqrt{\frac{k}{m}}$. This gives the differential equation $\ddot{x} + \omega^2 x = 0$. It's characteristic polynomial is $p(r) = r^2 + \omega^2 = 0 \quad \Rightarrow \quad r = \pm i\omega$. So all solutions to this homogeneous equation can be expressed as the span of $\left\{e^{i\omega t}, e^{-i\omega t}\right\}$, i.e. in the form $x(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$ where c_1, c_2 are <u>complex</u> constants. We would, of course, prefer to express solutions as real-valued functions. Using Euler's Formula, we could rewrite the solutions as $x(t) = c_1(\cos\omega t + i\sin\omega t) + c_2(\cos\omega t - i\sin\omega t) = (c_1 + c_2)\cos\omega t + i(c_1 - c_2)\sin\omega t$ and then hope that any given initial condition will produce real coefficients (they will). Another way to look at this is to note that since $e^{i\omega t} = \cos\omega t + i\sin\omega t$ and $e^{-i\omega t} = \cos\omega t - i\sin\omega t$ and we can also solve for $\cos\omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$ and $\sin\omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$, it must be the case that $\operatorname{Span}\left\{e^{i\omega t}, e^{-i\omega t}\right\} = \operatorname{Span}\left\{\cos\omega t, \sin\omega t\right\}$. That is, all solutions must be of the form $\boxed{x(t) = a\cos\omega t + b\sin\omega t}$. We also have the option of expressing this as $\boxed{x(t) = A\cos(\omega t - \phi)}$ where $A = \sqrt{a^2 + b^2}$ and $\tan\phi = \frac{b}{a}$.

Note: If we felt the urge to inquire whether the set $\left\{e^{i\omega t}, e^{-i\omega t}\right\}$ or the set $\left\{\cos\omega t, \sin\omega t\right\}$ were linearly independent solutions, the corresponding Wronskians would give either $\begin{vmatrix} e^{i\omega t} & e^{-i\omega t} \\ i\omega e^{i\omega t} & -i\omega e^{-i\omega t} \end{vmatrix} = -2i\omega \neq 0$ or $\begin{vmatrix} \cos\omega t & \sin\omega t \\ -\omega\sin\omega t & \omega\cos\omega t \end{vmatrix} = \omega(\cos^2\omega t + \sin^2\omega t) = \omega \neq 0$. They both provide a linearly independent spanning set for the solutions, i.e. a **basis** for the solutions (in linear algebra terms).

In the next lecture we'll look at the cases where there is friction and classify these as underdamped, overdamped, or critically damped. The underdamped case is characterized by decaying oscillatory solutions. The overdamped case is characterized by exponential decay. The critically damped case will be most interesting in terms of the concepts linear independence and spanning sets we just introduced. After that we'll move on to driven systems.

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