Concourse 18.03 – Lecture #6-7

This week we'll further explore the use of complex-valued functions as a tool for finding particular solutions to any linear ODE of the form $T(f)=g$ where the input $g(t)$ is any function of the form $g(t)=ke^{at}$ or $g(t)=ke^{at}\cos bt$ or $g(t)=ke^{at}\sin bt$ for various choices of the constants k,a,b. We'll also take a step back and look again at <u>autonomous systems</u>, i.e. ODEs of the form $\frac{dP}{dt}=F(P)$, by considering the <u>phase line</u>, the corresponding slope field, and by understanding the idea of <u>stability</u> in the vicinity of any equilibrium.

(First order) Linear response to exponential, sinusoidal inputs

Motivating example: Heating/cooling can be modeled by the ODE $\frac{dx}{dt}=k(y-x)$ where $x(t)$ measures the temperature inside some box, room, or other space, and where the outside temperature varies according to some prescribed function $y(t)$, with $k>0$ the coupling constant. This can also be written as $\boxed{\frac{dx}{dt}+kx=ky}$, so this can be thought of as a 1st order linear inhomogeneous differential equation with input $g(t)=k\,y(t)$.

Imagine a situation where the initial inside temperature is $x(0)=x_0$ and where the outside temperature varies sinusoidally according to $y(t)=A\cos\omega t$. Here ω is the frequency and the period is $T=\frac{2\pi}{\omega}$.

What do we expect will happen?

(a) The temperature variation (amplitude) inside will likely not be as great as the variation outside.

(b) Any initial temperature inside will be transient – as the system eventually takes over.

(c) The change in temperature inside will likely lag or be out of phase with the outside temperature (wine cellar effect)

(d) If the frequency ω is very small (slow change), we might expect the inside temperature to "keep up" with the outside temperature.

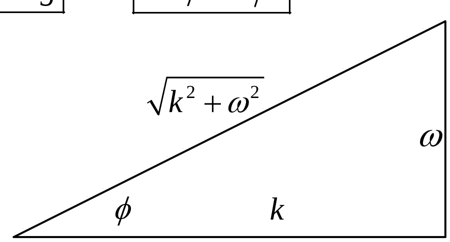
(e) If ω is very large (rapid oscillation of temperature), we expect that the inside temperature will have very small variation around the average temperature (which is 0 in this case).

To solve the given linear differential equation, we start by finding the homogeneous solutions. We rewrite $\frac{dx}{dt}+kx=0$ as $\frac{dx}{dt}=-kx$ and get $x_h(t)=ce^{-kt}$. For a particular solution, we could use undetermined coefficients and a solution of the form $x_p(t)=a\cos\omega t+b\sin\omega t$, but based on our expectations we might alternatively seek a solution of the form $x_p(t)=gA\cos(\omega t-\phi)$, where g as the ratio of response amplitude to input amplitude A. [This is equivalent to a solution of the form $x_p(t)=a\cos\omega t+b\sin\omega t$.] This ratio g is called the **gain**. We then substitute $x_p(t)=gA\cos(\omega t-\phi)$ into the original inhomogeneous ODE to determine g and ϕ.

We calculate $\frac{dx}{dt}+kx=-g\omega A\sin(\omega t-\phi)+kgA\cos(\omega t-\phi)=kA\cos\omega t$. To facilitate the determination of the unknowns g and ϕ, we rewrite $kA\cos\omega t=kA\cos(\omega t-\phi+\phi)=kA\cos\phi\cos(\omega t-\phi)-kA\sin\phi\sin(\omega t-\phi)$. So $\frac{dx}{dt}+kx=-g\omega A\sin(\omega t-\phi)+kgA\cos(\omega t-\phi)=kA\cos\phi\cos(\omega t-\phi)-kA\sin\phi\sin(\omega t-\phi)$. Equating coefficients gives $-g\omega A=-kA\sin\phi \;\Rightarrow\; \boxed{\sin\phi=g\omega/k}$, and $kgA=kA\cos\phi \;\Rightarrow\; \boxed{\cos\phi=g}$. So $\boxed{\tan\phi=\omega/k}$

This is most easily pictured with a right triangle as shown.

From this we see that $\boxed{\tan\phi=\frac{\omega}{k}}$ and $\boxed{g=\frac{k}{\sqrt{k^2+\omega^2}}}$.



So, the particular solution is $x_p(t)=gA\cos(\omega t-\phi)$ with these values for the gain g and the phase angle ϕ, and the general solution is therefore $x(t)=ce^{-kt}+gA\cos(\omega t-\phi)$. <u>Does this match with our expectations</u>?

Notes: (1) When the frequency ω is small (slow change), the gain g will be close to $1 = 100\%$, i.e. the inside temperature will vary almost as much as the outside temperature, and the lag will be close to 0 (temperature inside will "keep up" with the outside temperature change).

(2) When the frequency ω is large (rapid change), the gain g will be close to 0, so the inside temperature will have very small variation around the average temperature of 0. It will also be the case that the lag will approach 90°, but this will likely go unnoticed due to the minimal temperature variation.

(3) The initial temperature inside will determine the constant c in the exponentially decaying (transient) term, and this term will become negligible over time.

Another approach to finding a solution is to introduce complex-valued functions. For this we'll actually be solving two differential equations simultaneously. In addition to the ODE $\frac{dx}{dt} + kx = kA\cos\omega t$, let's also consider the ODE $\frac{dy}{dt} + ky = kA\sin\omega t$. If we let $z(t) = x(t) + i\,y(t)$, then $\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$, so we'll have $\frac{dz}{dt} + kz = \left(\frac{dx}{dt} + i\frac{dy}{dt}\right) + k\left(x + iy\right) = \left(\frac{dx}{dt} + kx\right) + i\left(\frac{dy}{dt} + ky\right) = kA(\cos\omega t + i\sin\omega t) = kAe^{i\omega t}$, using Euler's formula.

This gives the complex ODE $\frac{dz}{dt} + kz = kAe^{i\omega t}$ where now the right-hand-side is now an exponential function. We will soon develop a handy tool called the Exponential Response Formula (ERF) for handling similar linear ODE's of any order, but for now we can solve this directly using undetermined coefficients. The homogeneous solutions will again be of the form $z_h(t) = ce^{-kt}$, but we must understand the constant c to be an arbitrary complex constant, i.e. $c = c_1 + ic_2$. The homogeneous solutions may this be written as $z_h(t) = c_1e^{-kt} + ic_2e^{-kt}$.

For a particular solution, we try $z_p(t) = GAe^{i\omega t}$ where G is a complex constant called the **complex gain**. Differentiation and substitution into the ODE gives $\frac{dz}{dt} + kz = GAi\omega e^{i\omega t} + kGAe^{i\omega t} = GA(k + i\omega)e^{i\omega t} = kAe^{i\omega t}$, so we must have $G(k + i\omega) = k$ or $G = \frac{k}{k+i\omega}$. If we refer to the triangle from before and write in polar form $k + i\omega = \sqrt{k^2 + \omega^2}\, e^{i\phi}$, we'll have $G = \frac{k}{\sqrt{k^2+\omega^2}\, e^{i\phi}}$, and the particular solution will be

$z_p(t) = \frac{kA}{\sqrt{k^2+\omega^2}\, e^{i\phi}} e^{i\omega t} = \frac{kA}{\sqrt{k^2+\omega^2}} e^{i(\omega t - \phi)}$. Note that $|G| = \left|\frac{k}{\sqrt{k^2+\omega^2}\, e^{i\phi}}\right| = \frac{k}{\sqrt{k^2+\omega^2}} = g$ is the gain, and we can write

$z_p(t) = \frac{kA}{\sqrt{k^2+\omega^2}} e^{i(\omega t-\phi)} = \left[\frac{kA}{\sqrt{k^2+\omega^2}}\cos(\omega t - \phi)\right] + i\left[\frac{kA}{\sqrt{k^2+\omega^2}}\sin(\omega t - \phi)\right]$.

So we have $z(t) = \left[c_1e^{-kt} + \frac{kA}{\sqrt{k^2+\omega^2}}\cos(\omega t - \phi)\right] + i\left[c_2e^{-kt} + \frac{kA}{\sqrt{k^2+\omega^2}}\sin(\omega t - \phi)\right] = x(t) + i\,y(t)$ as the general solution. This individually gives solutions $\boxed{x(t) = c_1e^{-kt} + \frac{kA}{\sqrt{k^2+\omega^2}}\cos(\omega t - \phi)}$ to the first ODE and $\boxed{y(t) = c_2e^{-kt} + \frac{kA}{\sqrt{k^2+\omega^2}}\sin(\omega t - \phi)}$ to the second ODE, and the solution to the first ODE is consistent with what we derived previously. Here $g = \frac{k}{\sqrt{k^2+\omega^2}}$ is the gain and $\tan\phi = \frac{\omega}{k}$ determines the lag. This method involving complex solutions is especially appropriate when considering gain and lag in the solution of higher order linear ODEs when the inhomogeneity is of the form $q(t) = ke^{at}\cos\omega t$ or $q(t) = ke^{at}\sin\omega t$.

Engineers often plot the gain and lag as functions of the input frequency ω. The plots of $\log\left[g(\omega)\right]$ vs. $\log(\omega)$ and $-\phi(\omega)$ vs. $\log(\omega)$ are known as **Bode plots**. They measure the response to a given signal.

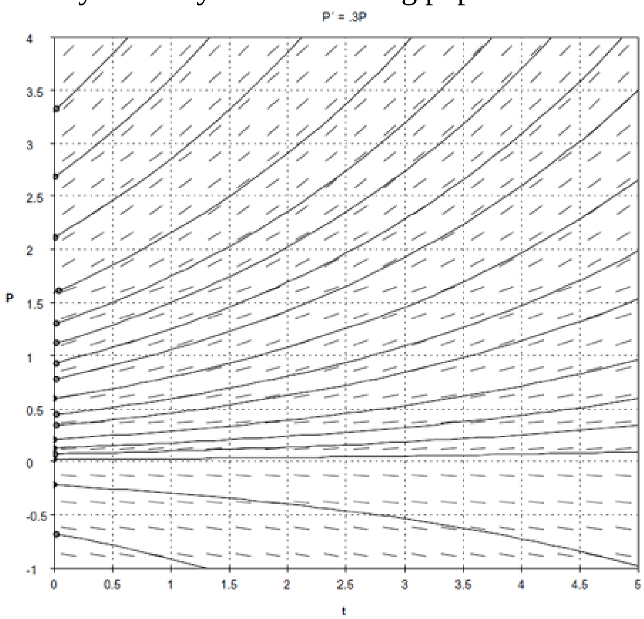
Autonomous differential equations

Definition: An **first order autonomous differential equation** is an ODE of the form $\frac{dx}{dt} = F(x)$, i.e. an ODE where the rate $\frac{dx}{dt}$ depends only on the value of *x*. If *t* represents time, this means that the rate of change is time-independent.

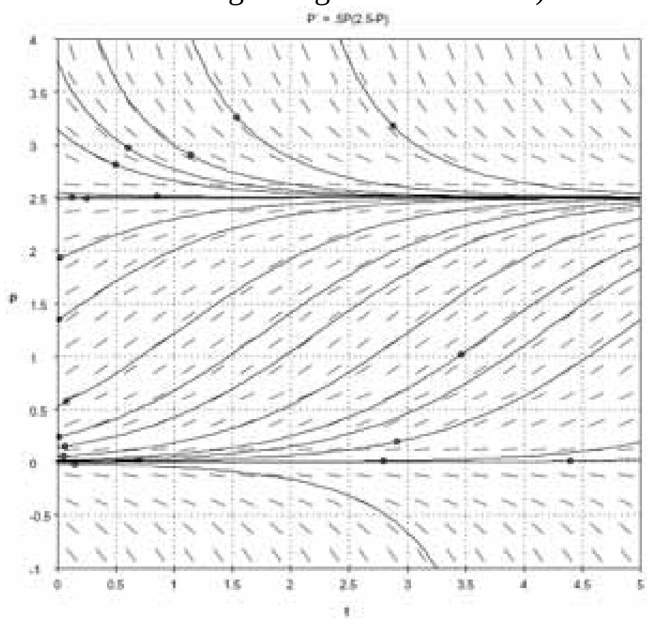
If we draw the slope field corresponding to an autonomous equation, the slopes will be constant horizontally but may vary vertically. Two familiar autonomous ODE's are:

(a) Natural (unrestricted) growth: $\boxed{\frac{dx}{dt} = kx}$ (exponential growth for $k > 0$, exponential decay for $k < 0$)

(b) Logistic growth: $\boxed{\frac{dx}{dt} = kx(1-\frac{x}{L})}$ (*L* is the "carrying capacity", the relative growth rate $\frac{1}{x}\frac{dx}{dt} = k(1-\frac{x}{L})$ decays linearly with increasing population with rate 0 when $x = L$ and negative growth for $x > L$)



Natural (exponential) growth: $\frac{dx}{dt} = kx$



Logistic growth: $\frac{dx}{dt} = kx(1-\frac{x}{L})$

Even if we can solve an autonomous differential equation analytically to get a formula for the solutions, it is often more important to understand the solutions qualitatively.

Definition: Given an autonomous differential equation $\frac{dx}{dt} = F(x)$, we call a point x_0 an **equilibrium** if $F(x_0) = 0$. The constant solution $x(t) = x_0$ will be a solution to the differential equation with initial condition $x(0) = x_0$.

As we can see in the illustrations above, some equilibria are such that nearby solutions converge toward the equilibrium and other equilibria are such that nearby solutions diverge away from the equilibrium.

Definition: If x_0 is an equilibrium of $\frac{dx}{dt} = F(x)$ and if for all initial conditions in some interval around x_0 the solutions $x(t)$ are such that $\lim_{t\to\infty}\left[x(t)\right] = x_0$, then we call x_0 a **stable equilibrium**. Otherwise we call it an **unstable equilibrium**. However, we usually consider an unstable equilibrium to be such that nearby solutions diverge away from the equilibrium. If we draw only the *x*-axis and indicate equilibria as points with arrows indicating the direction of nearby solutions, we refer to this as the **phase line**.

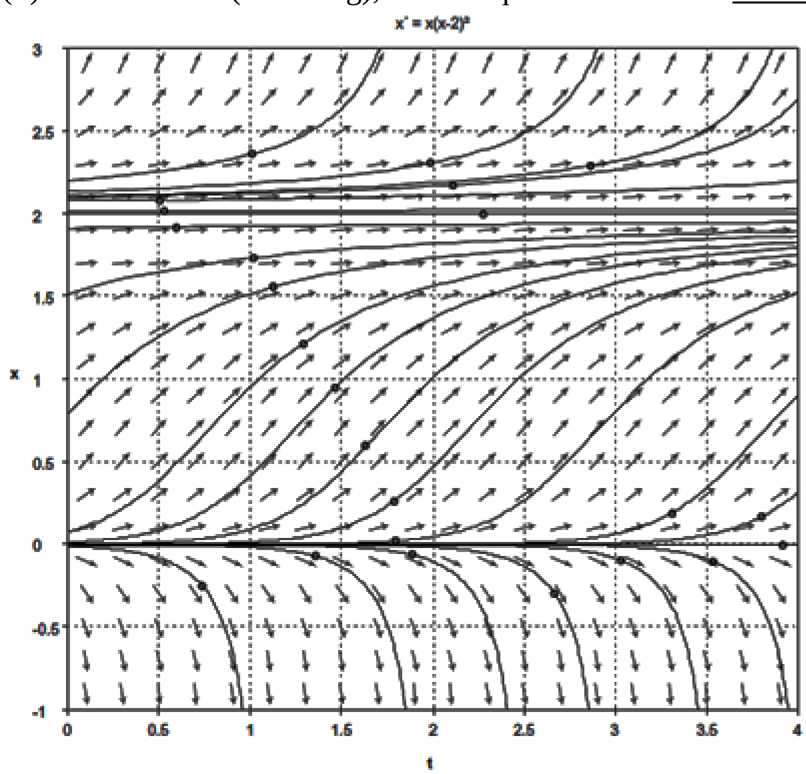
There's a simple **derivative test** for distinguishing stable and unstable equilibria. Suppose x_0 is an equilibrium for the differential equation $\frac{dx}{dt} = F(x)$ and that $F(x)$ is differentiable at x_0. We learned in Calculus about linear approximation, and in the vicinity of x_0 we'll have $F(x) \cong F(x_0) + F'(x_0)(x - x_0) = F'(x_0)(x - x_0)$

because $F(x_0)=0$. We also know that if we let $u=(x-x_0)$, then $\frac{du}{dt}=\frac{d}{dt}(x-x_0)=\frac{dx}{dt}$, so we'll have $\frac{du}{dt}\cong F'(x_0)(x-x_0)=F'(x_0)u$. The differential equation $\frac{du}{dt}=F'(x_0)u$ yields growth (away from $u=0$ or $x=x_0$) if $F'(x_0)>0$, and decay (toward $u=0$ or $x=x_0$) if $F'(x_0)<0$. This enables us to distinguish unstable and stable equilibria. In the case where $F'(x_0)=0$, we'll have to look at the slope field or use similar analysis.

Note: It may happen that on one side of an equilibrium nearby solutions converge toward the equilibrium but on the other side they diverge away from the equilibrium. In this case we would call the equilibrium **semistable**.

Example: Determine the equilibria of the differential equation $\frac{dx}{dt}=x(x-2)^2$ and classify their stability.

Solution: The equilibria will be where $F(x)=x(x-2)^2=0$, i.e. at $x=0$ and at $x=2$. The derivative gives $F'(x)=(x-2)(3x-2)$. We have $F'(0)=4>0$ so this equilibrium will be unstable. On the other hand, $F'(2)=0$ so we must use other means to determine the stability of this equilibrium. Note that $F'(x)>0$ for $x>2$ (repelling) and $F'(x)<0$ for $x<2$ (attracting), so this equilibrium will be semistable.



Analytic solution of the logistic equation

The logistic equation is $\frac{dx}{dt}=kx(1-\frac{x}{L})$ where $k>0$ is constant. It has an unstable equilibrium at $x=0$ and a stable equilibrium at $x=L$ (the carrying capacity). Suppose $x(0)=x_0$ is the initial condition. We can write $\frac{dx}{x(1-\frac{x}{L})}=kdt$ and $\int\frac{dx}{x(1-\frac{x}{L})}=\int kdt=kt+C$. The integral on the left is done using partial fractions. Specifically, $\frac{1}{x(1-\frac{x}{L})}=\frac{L}{x(L-x)}=\frac{A}{x}+\frac{B}{L-x} \quad\Rightarrow\quad L=A(L-x)+Bx$. Choosing $x=0$ gives $AL=L$ or $A=1$. Choosing $x=L$ gives $BL=L$ or $B=1$. So $\frac{1}{x(1-\frac{x}{L})}=\frac{1}{x}+\frac{1}{L-x} \quad\Rightarrow\quad \int\frac{dx}{x(1-\frac{x}{L})}=\int\left(\frac{1}{x}+\frac{1}{L-x}\right)dx=\ln|x|-\ln|L-x|=\ln\left|\frac{x}{L-x}\right|$.

So $\ln\left|\frac{x}{L-x}\right|=kt+C \quad\Rightarrow\quad \frac{x}{L-x}=Ae^{kt} \quad\Rightarrow\quad x=LAe^{kt}-Axe^{kt} \quad\Rightarrow\quad x(1+Ae^{kt})= \quad\Rightarrow\quad x(t)=\frac{LAe^{kt}}{1+Ae^{kt}}$.

The initial condition gives $x(0)=\frac{LA}{1+A}=x_0 \quad\Rightarrow\quad LA=x_0+Ax_0 \quad\Rightarrow\quad A(L-x_0)=x_0 \quad\Rightarrow\quad A=\frac{x_0}{L-x_0}$.

So $x(t)=\dfrac{L\left(\dfrac{x_0}{L-x_0}\right)e^{kt}}{1+\left(\dfrac{x_0}{L-x_0}\right)e^{kt}}=\dfrac{Lx_0e^{kt}}{(L-x_0)+x_0e^{kt}}=\dfrac{Lx_0}{x_0+(L-x_0)e^{-kt}}$. So the solution is $\boxed{x(t)=\dfrac{Lx_0}{x_0+(L-x_0)e^{-kt}}}$.

Note, in particular, that $\lim\limits_{t\to\infty}\left[x(t)\right]=\lim\limits_{t\to\infty}\left[\dfrac{Lx_0}{x_0+(L-x_0)e^{-kt}}\right]=L$, as expected.

Example: Suppose population growth is governed by the logistic differential equation $\frac{dx}{dt}=kx(1-\frac{x}{L})$ with $k=1$ and carrying capacity $L=1000$. Further suppose that the initial population is $x(0)=100$. The analytic solution will then be $x(t)=\dfrac{100000}{100+900e^{-t}}=\dfrac{1000}{1+9e^{-t}}$. If we would like to know when the population will reach 500, we have $x(t)=\dfrac{1000}{1+9e^{-t}}=500 \Rightarrow 1000=500+4500e^{-t} \Rightarrow e^t=9 \Rightarrow t=\ln 9\cong 2.197$.

If we ask when the population will reach 990, we have

$x(t)=\dfrac{1000}{1+9e^{-t}}=990 \Rightarrow 1000=990+8910e^{-t} \Rightarrow e^t=891 \Rightarrow t=\ln 891\cong 6.792$.

We occupied the remainder of Lecture #7 with questions from the Practice Exam.

Notes by Robert Winters