Concourse 18.03 – Lecture #5

Today's topics include another method for finding particular solutions for linear ODEs – variation of parameters – and the introduction of complex numbers and related facts to reformulate some of the methods involving exponential and sinusoidal inputs.

Methods we've seen so far:
Separation of variables
Integrating factors for solving 1st order linear ODEs
Linearity – homogeneous and particular solutions for linear ODEs
Method of undetermined coefficients for finding particular solutions to linear ODEs

---

Variation of parameters
Another useful method for finding a particular solution to a linear ODE is the take the homogeneous solutions that you've presumably already found and "vary the parameters." This method can be formulated for nth order linear ODEs (and we'll do that eventually), but for now we'll formulate the method for 1st order linear ODEs.

Suppose we are trying to solve the linear ODE $\frac{dy}{dx} + p(x)\cdot y = q(x)$ where $p(x), q(x)$ are functions of the independent variable *x*, and that we have already solved the homogeneous equation $\frac{dy}{dx} + p(x)\cdot y = 0$ to find the homogeneous solutions $y_h(x)$. This equation is separable and can, in principle, always be solved to give $y_h(x) = Ae^{-\int p(x)dx}$. The basic idea is to treat the scalar A as variable.

If we write $y(x) = v(x)y_h(x)$ where $y_h(x)$ as the basic homogeneous solution, we can then calculate that $\frac{dy}{dx} = v(x)\frac{dy_h}{dx} + v'(x)y_h(x)$ and substitute into the ODE to get:

$$v(x)\frac{dy_h}{dx} + v'(x)y_h(x) + p(x)v(x)y_h(x) = v(x)\left(\frac{dy_h}{dx} + p(x)y_h(x)\right) + v'(x)y_h(x) = q(x)$$

Note that since $y_h(x)$ is a solution to the homogeneous equation, the expression in parentheses vanishes. So the resulting equation becomes $v'(x)y_h(x) = q(x)$. This is, in principle, easily solved by writing $v'(x) = \frac{q(x)}{y_h(x)}$ and integrating to get $\boxed{v(x) = \int \frac{q(x)}{y_h(x)}dx}$. We then have the particular solution $y_p(x) = v(x)y_h(x)$.

Example #1: Find the general solution of the 1st order linear ODE $\frac{dy}{dx} + \frac{5}{x}y = 7x$.

Solution: The homogeneous equation $\frac{dy}{dx} + \frac{5}{x}y = 0$ gives:

$$\frac{dy}{dx} = -\frac{5}{x}y \quad\Rightarrow\quad \frac{dy}{y} = -\frac{5}{x}dx \quad\Rightarrow\quad \int\frac{dy}{y} = -\int\frac{5}{x}dx \quad\Rightarrow\quad \ln|y| = -5\ln|x| + C \quad\Rightarrow\quad y_h(x) = Ax^{-5}$$

So we take $y_h(x) = x^{-5} = \frac{1}{x^5}$ for the purpose of doing variation of parameters to find a particular solution. With $q(x) = 7x$, the method as described above gives $v(x) = \int \frac{7x}{x^{-5}}dx = \int 7x^6 dx = x^7$. [Note that we don't add an arbitrary constant because we're only trying to find one particular solution.]

So $y_p(x) = v(x)y_h(x) = x^7 \cdot x^{-5} = x^2$. The general solution is therefore $\boxed{y(x) = Ax^{-5} + x^2}$ where *A* is an arbitrary constant.

 revised February 19, 2020

Example #2 (sinusoidal input): Find the general solution to the ODE $\frac{dx}{dt}+2x=\cos 3t$. [This is the same problem we solved in the previous lecture.]

Solution: Last time we solved the homogeneous ODE to get $x_h(t)=ce^{-2t}$. If we use $x_h(t)=e^{-2t}$ and $q(t)=\cos 3t$ for the variation of parameters, we get $v(t)=\int \frac{q(t)}{x_h(t)}\,dt=\int \frac{\cos 3t}{e^{-2t}}\,dt=\int e^{2t}\cos 3t\,dt$.

The integral is found using integration by parts (twice) and some algebra. As a reminder of integration methods, the calculation would go something like this:

$$I=\int e^{2t}\cos 3t\,dt=\tfrac{1}{3}e^{2t}\sin 3t-\tfrac{2}{3}\int e^{2t}\sin 3t\,dt=\tfrac{1}{3}e^{2t}\sin 3t-\tfrac{2}{3}\left[-\tfrac{1}{3}e^{2t}\cos 3t+\tfrac{2}{3}\int e^{2t}\cos 3t\,dt\right]$$

$$=e^{2t}\left(\tfrac{1}{3}\sin 3t+\tfrac{2}{9}\cos 3t\right)-\tfrac{4}{9}I \;\Rightarrow\; \tfrac{13}{9}I=e^{2t}\left(\tfrac{1}{3}\sin 3t+\tfrac{2}{9}\cos 3t\right) \;\Rightarrow\; I=\boxed{\int e^{2t}\cos 3t\,dt=e^{2t}\left(\tfrac{2}{13}\cos 3t+\tfrac{3}{13}\sin 3t\right)}$$

So the particular solution is $x_p(t)=\left(e^{2t}\left(\tfrac{2}{13}\cos 3t+\tfrac{3}{13}\sin 3t\right)\right)e^{-2t}=\tfrac{2}{13}\cos 3t+\tfrac{3}{13}\sin 3t$ and the general solution is then $\boxed{x(t)=ce^{-2t}+\tfrac{2}{13}\cos 3t+\tfrac{3}{13}\sin 3t}$ where c is an arbitrary constant to be determined by initial conditions. You can check that this coincides with the solution we derived last time via other methods. We also have the option of putting this in the form $\boxed{x(t)=ce^{-2t}+\tfrac{1}{\sqrt{13}}\cos(3t-\phi_0)}$ where $\phi_0\cong 56.31°$ as we showed last time.

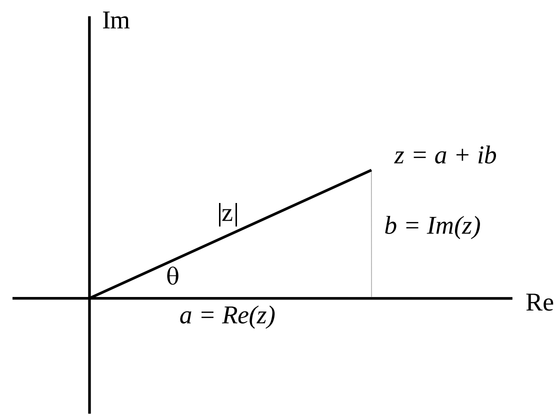
Complex variable methods for working with sinusoidal and exponential inputs

The calculation above might lead you to believe that any time we're dealing with a linear ODE of the form $T(f)=g$ where the input is sinusoidal we should expect involved integral calculations. Indeed, we might consider inputs of the form $g(t)=ke^{at}$ or $g(t)=ke^{at}\cos bt$ or $g(t)=ke^{at}\sin bt$ for various choices of the constants k,a,b.

Somewhere in your mathematical history you most likely learned a few things about **complex numbers**. We initially express complex numbers in the rectangular form $z=a+ib$ where $i^2=-1$. Complex numbers can be viewed in vector-like terms in the complex plane as shown in the diagram. We define:

modulus (z) = mod (z) = $|z|$ = $\sqrt{a^2+b^2}$

argument (z) = arg (z) = $\theta=\tan^{-1}\left(\frac{b}{a}\right)$.



We add complex numbers by adding their respective real and imaginary parts, in much the same way as vector addition is defined. We multiply complex numbers via the distributive law and the fact that $i^2=-1$. For example:

$$(3+2i)(-1-4i)=-3-2i-12i-8i^2=-3-14i+8=5-14i.$$

If we note that $a=|z|\cos\theta$ and $b=|z|\sin\theta$, then we can write $z=a+bi=|z|(\cos\theta+i\sin\theta)$. There's a simpler way to express this using <u>Euler's formula</u>. The Maclaurin series for e^t, $\cos t$, and $\sin t$ are:

$$\begin{Bmatrix} e^t=1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots \\ \cos t=1-\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots \\ \sin t=t-\frac{t^3}{3!}+\frac{t^5}{5!}-\cdots \end{Bmatrix}.$$

If we formally replace t by it and use the usual algebra rules, we get that:

$$e^{it}=1+it+\frac{(it)^2}{2!}+\frac{(it)^3}{3!}+\frac{(it)^4}{4!}+\cdots=\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\cdots\right)+i\left(t-\frac{t^3}{3!}+\cdots\right)=\cos t+i\sin t$$

That is, $\boxed{e^{it} = \cos t + i\sin t}$ [**Euler's Formula**]

A curious corollary of this is Euler's Identity: $e^{i\pi} = -1$.

Using Euler's Formula, we can express any complex number as $z = a + bi = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$ where $|z|$ is the modulus and θ is the argument of the complex number. This **polar form** allows us to understand the multiplication of complex numbers in very geometric terms. That is, if $z_1 = |z_1|e^{i\theta_1}$ and $z_2 = |z_2|e^{i\theta_2}$ are two complex numbers, their product is $z_1 z_2 = |z_1||z_2|e^{i\theta_1}e^{i\theta_2} = |z_1||z_2|e^{i(\theta_1+\theta_2)}$. That is, the modulus of the product is given by $|z_1 z_2| = |z_1||z_2|$ and the argument of the product is given by $Arg(z_1 z_2) = \theta_1 + \theta_2 = Arg(z_1) + Arg(z_2)$. When we multiply complex numbers, we multiply the moduli and we add the arguments.
As a special case, note that the complex number i has modulus 1 and argument $\pi/2 = 90°$. So i^2 should have modulus 1 and argument $\pi = 180°$, and this does indeed correspond to -1.

Perhaps more interesting is what this tells us about the "**roots of unity**". If we seek solutions to the equation $z^n = 1$ or, equivalently, $z^n - 1 = 0$, we know that $z = 1$ is a solution, but what are the other solutions? One way to approach this might be via factoring, i.e. $z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \cdots + z + 1) = 0$ and we'd be seeking a factorization of $z^{n-1} + z^{n-2} + \cdots + z + 1 = 0$. If, instead, we think of this geometrically, it should be pretty clear that any such root would have to have modulus 1 (so it would lie on the unit circle in the complex plane) and it's argument θ would have to be such that $n\theta = 2\pi k$ for some integer *k*. Any such number must be of the form $z = e^{i(2\pi k/n)}$, and these consist of *n* points evenly distributed on the unit circle including $z = 1$. For example , the solutions to $z^3 = 1$ would be $\left\{1, e^{i(2\pi/3)}, e^{i(4\pi/3)}\right\}$, i.e. $\left\{1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}\right\}$.

Definition: The **complex conjugate** of $z = a + ib$ is defined to be $\bar{z} = a - ib$. In the complex plane, *z* and $\bar{z}$ are reflections of each other across the real axis. It's not hard to show that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.

When factoring polynomials with real coefficients, the Fundamental Theorem of Algebra and the Quadratic Formula guarantee that any complex roots must come in complex conjugate pairs.

A little more trigonometry

We can use Euler's formula to produce a quick derivation of the sum of angle formulas for both the sine and cosine functions. We have:

$$e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi).$$

So, since $e^{i(\theta+\phi)} = \cos(\theta+\phi) + i\sin(\theta+\phi)$, comparing the real parts and the imaginary parts give that:

$$\boxed{\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi} \text{ and } \boxed{\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi}.$$

Application to integration

We can actually find the integrals $\int e^{at}\cos bt\,dt$ and $\int e^{at}\sin bt\,dt$ simultaneously using complex numbers.

If we write $e^{ibt} = \cos bt + i\sin bt$, then $e^{(a+ib)t} = e^{at}e^{ibt} = e^{at}\cos bt + ie^{at}\sin bt$.

Integration acts linearly, and if we extend this to complex-valued functions, we have that:

$$\int e^{(a+ib)t}dt = \int e^{at}\cos bt\,dt + i\int e^{at}\sin bt\,dt.$$

Exponential functions are easy to integrate (even when we extend to complex-valued exponential functions), and we calculate that $\int e^{(a+ib)t}dt = \frac{1}{a+ib}e^{(a+ib)t}$. We can proceed several ways here, but for the purpose of calculating these integrals, let's get rid of the complex denominator by multiplying both numerator and denominator by its complex conjugate (and use the fact that $z\overline{z} = (a+ib)(a-ib) = a^2 + b^2 = |z|^2$). We get:

$$\int e^{(a+ib)t}dt = \tfrac{1}{a+ib}e^{(a+ib)t} = \tfrac{a-ib}{a^2+b^2}e^{at}e^{ibt} = \tfrac{1}{a^2+b^2}e^{at}(a-ib)(\cos bt + i\sin bt)$$
$$= \tfrac{1}{a^2+b^2}e^{at}\left[(a\cos bt + b\sin bt) + i(-b\cos bt + a\sin bt)\right]$$

If we compare this with $\int e^{(a+ib)t}dt = \int e^{at}\cos bt\,dt + i\int e^{at}\sin bt\,dt$, we see that:

$$\boxed{\int e^{at}\cos bt\,dt = \tfrac{1}{a^2+b^2}e^{at}(a\cos bt + b\sin bt)} \text{ and } \boxed{\int e^{at}\sin bt\,dt = \tfrac{1}{a^2+b^2}e^{at}(-b\cos bt + a\sin bt)}.$$

If we were to apply this to the integral calculated earlier using integration by parts, we'd get that:

$$\boxed{\int e^{2t}\cos 3t\,dt = \tfrac{1}{13}e^{2t}\left(2\cos 3t + 3\sin 3t\right) + C}$$

This agrees with our previous result.

Next time we'll apply these methods involving complex-valued functions to discover a remarkably simple way of finding particular solutions to any linear ODE of the form $T(f) = g$ where the input $g(t)$ is any function of the form $g(t) = ke^{at}$ or $g(t) = ke^{at}\cos bt$ or $g(t) = ke^{at}\sin bt$ for various choices of the constants k, a, b .

We'll also take a step back and look at autonomous systems in general, i.e. ODEs of the form $\frac{dP}{dt} = F(P)$, by considering the phase line, the corresponding slope field, and by understanding the idea of stability in the vicinity of any equilibrium.

Notes by Robert Winters