Concourse 18.03 – Lecture #4

The central theme for today is the **Input-Response** formulation for linear ODEs.

A linear ODE of the form
\[ x^{(n)}(t) + p_{n-1}(t)x^{(n-1)}(t) + \cdots + p_1(t)x'(t) + p_0(t)x(t) = q(t) \]
where \( p_{n-1}(t), \ldots, p_1(t), p_0(t), q(t) \) are functions of the independent variable \( t \) can be expressed in the form
\[ T(x(t)) = g(t) \]
where \( T \) is a linear operator of the form
\[ T = \frac{d^n}{dt^n} + p_{n-1}(t) \frac{d^{n-1}}{dt^{n-1}} + \cdots + p_1(t) \frac{d}{dt} + p_0(t) \cdot \]
The last term refers to multiplication by \( p_0(t) \). A useful way of formulating such an ODE is to think of the left-hand-side as corresponding to “the system” and the inhomogeneity \( q(t) \) on the right-hand-side as corresponding to the “input signal” or, more simply, the “input.” The general solution of the ODE is then referred to as the “output signal” or “response.” Some motivating examples are in order.

**Banking:** If we let \( x = x(t) \) represent how much money (in dollars) we have in a bank account after \( t \) years with a fixed interest rate \( I \), the simple model for this is \( \frac{dx}{dt} = Ix \) and we have solved this to get \( x(t) = Ae^{bt} \). If we have only the initial deposit \( x(0) = x_0 \), then we’ll have \( x(t) = x_0e^{bt} \). Note, however, that we can write the ODE as \( \frac{dx}{dt} - Ix = 0 \), a homogeneous 1st order linear ODE. This corresponds to the situation where you make your deposit and then go home and let the system grow your money without further intervention.

Now let’s suppose that you make deposits and withdrawals according to some function \( q(t) \) (in dollars/year). If we add this rate into our model, we have \( \frac{dx}{dt} = Ix + q(t) \), or \( \frac{dx}{dt} - Ix = q(t) \). Note how this intervention (or input) corresponds to the inhomogeneity of this linear ODE. The “system” will carry on as before but will be subject to the input associated with the deposits and withdrawals. The “response” to all this internal and external activity will be the output \( x(t) \), i.e. how much money you’ll have in the bank at any given time.

**Newton’s Law of Cooling (diffusion):** Suppose we have an enclosed space such as a building or a cooler chest and that the temperature at any given time \( t \) in the interior space is measured as some function \( x(t) \) and that the initial temperature is \( x(0) = x_0 \). If the outside temperature is given by some function \( y(t) \) (possibly constant or possibly variable), then we might expect the interior temperature to change depending on the quality of the insulation and on the difference between outside and inside temperatures. That is, the rate of change of temperature might be modeled as \( \frac{dx}{dt} = F(y - x) \) for some function \( F \). We would expect that when \( y > x \) the temperature would increase, i.e. that \( \frac{dx}{dt} > 0 \); when \( y < x \) the temperature would decrease, i.e. that \( \frac{dx}{dt} < 0 \); and that when the outside temperatures are the same there would be no change in temperature, i.e. \( \frac{dx}{dt} = 0 \). The simplest model for this would be \( \frac{dx}{dt} = k(y - x) \) for some positive constant \( k \) (called the coupling constant) that depends on the level of insulation. We can rewrite this as \( \frac{dx}{dt} + kx = ky \). In this form, we can think of the homogeneous equation \( \frac{dx}{dt} + kx = 0 \) as representing how this system would be governed if the outside temperature remained constant (at 0) and the interior temperature gradually rose or fell to that level. The inhomogeneous equation \( \frac{dx}{dt} + kx(t) = k \cdot y(t) \) would govern how the interior temperature would respond to the input \( y(t) \) (or \( ky(t) \)) in the case where the outside temperature varied according to some known pattern, e.g. the sinusoidal temperature change that might be associated with either a day/night cycle or seasonal cycle, or perhaps some other temperature variation.

**Hooke’s Law:** A simple model for a frictionless mass-spring system is given by Hooke’s Law \( F = -kx \) where \( F \) represents an applied force, \( x \) represents the displacement of the mass from the equilibrium position, and
where \( k \) is the spring constant that corresponds to the stiffness of the spring. If we combine this with Newton’s 2nd Law that \( F = ma \) where \( m \) is the mass, \( v = \frac{dx}{dt} \) is the velocity, and \( a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \) is the acceleration of the mass, we have \( ma = -kx \) or \( m \frac{d^2x}{dt^2} = -kx \). We can rewrite this as \( \frac{d^2x}{dt^2} + \frac{k}{m} x = 0 \). If these is some friction in the system, a simple model suggests that this friction would grow proportionally to the velocity, i.e. there would be an additional force \( F_f = -cv \) opposing the motion. The revised equation becomes \( m \frac{d^2x}{dt^2} = -kx - cv \) or \( \frac{d^2x}{dt^2} + \frac{c}{m} \frac{dx}{dt} + \frac{k}{m} x = 0 \). Physicists often favor the “dot notation” for time derivatives with \( \dot{x} = \frac{dx}{dt} \) and \( \ddot{x} = \frac{d^2x}{dt^2} \), so the equation may also be expressed as \( \ddot{x} + \frac{c}{m} \dot{x} + \frac{k}{m} x = 0 \).

Now imagine that you mess with this spring system by “driving” the system with an additional external force \( E(t) \). The model might then look like \( m \frac{d^2x}{dt^2} = -kx - cv + E(t) \) and if we write \( E(t) = mq(t) \) for simplicity, the ODE becomes \( \frac{d^2x}{dt^2} + \frac{c}{m} \frac{dx}{dt} + \frac{k}{m} x = \frac{E(t)}{m} = q(t) \) or \( \ddot{x} + \frac{c}{m} \dot{x} + \frac{k}{m} x = q(t) \). Once again, the inhomogeneity corresponds to the external input imposed on the system, and the homogeneous ODE would govern how the system would evolve without this intervention.

As we’ve seen previously, a good approach to solving all of these linear ODEs is to use linear methods that involve finding all homogeneous solutions (the system), finding one particular solution, and combining these to determine the overall response \( x(t) \).

**Example (diffusion):** Suppose a closed container has an initial interior temperature of \( 32^\circ F \) at 10:00am and that the outside temperature (also in \( ^\circ F \)) rises steadily according to \( y(t) = 60 + 6t \) where time \( t \) is measured in hours. Further suppose that Newton’s Law of Cooling applies where the coupling constant is \( k = \frac{1}{3} \), (a) How will the interior temperature vary in time, and (b) at what time will the interior temperature reach \( 60^\circ F \)?

**Solution:** The temperature will be governed by \( \frac{dx}{dt} = \frac{1}{3} (y-x) \) or \( \frac{dx}{dt} + \frac{1}{3} x = \frac{1}{3} y(t) = \frac{1}{3} (60 + 6t) = 20 + 2t \), so the inhomogeneous ODE is \( \frac{dx}{dt} + \frac{1}{3} x = 20 + 2t \). This can be solved using an integrating factor, but let’s use linearity.

1. The homogeneous equation \( \frac{dx}{dt} + \frac{1}{3} x = 0 \) easily yields the solutions of the form \( x_h(t) = ce^{\frac{-t}{3}} \). It’s worth noting that over time any such homogeneous solution will tend toward 0 and become negligible. For this reason we often refer to this as a **transient**. In the short term it may be relevant, but in the long term it is not.

2. We can use undetermined coefficients to find a particular solution. The nature of the inhomogeneity \( q(t) = 20 + 2t \) suggests that we seek a solution of the form \( x_p(t) = A + Bt \). We have \( \frac{dx}{dt} (t) = B \), so we must have \( B + \frac{1}{3} (A + Bt) = (B + \frac{1}{3} A) + \frac{1}{3} Bt = 20 + 2t \), so \( B + \frac{1}{3} A = 20 \) and \( \frac{1}{3} B = 2 \). This gives \( B = 6 \) and \( A = 42 \), so \( x_p(t) = 42 + 6t \). Once the transients have become negligible, this is all that will remain. For this reason we might refer to this as the “steady state” solution.

3. The general solution is \( x(t) = x_h(t) + x_p(t) = ce^{\frac{-t}{3}} + 42 + 6t \). If we substitute the initial condition \( x(0) = 32 \), we have \( x(0) = c + 42 = 32 \), so \( c = -10 \) and \( x(t) = 42 + 6t - 10e^{\frac{-t}{3}} \). Note that eventually the interior temperature will be rising at the same rate as the outside temperature but always \( 18^\circ F \) cooler.

The interior temperature will reach \( 60^\circ F \) at a time \( T \) when \( 42 + 6T - 10e^{\frac{-T}{3}} = 60 \) or \( 6T - 10e^{\frac{-T}{3}} = 18 \). This cannot be solved algebraically, but it’s easy to get a numerical solution using a graphing calculator and the **trace** function. It gives a time \( T \approx 3.33 \approx 3 \) hrs, 20 min, i.e. about 1:20pm.

**Example #2 (exponential input):** Solve the initial value problem \( \frac{d^2x}{dt^2} + 3 \frac{dx}{dt} + 2x = e^t \), \( x(0) = 4 \), \( x'(0) = 2 \).
Solution: This ODE is of the type we might expect from a mass-spring system, though the external driving force is not especially realistic (relentlessly exponential in a single direction). It is nonetheless good for illustrating the methods, and the exponential input will be very relevant in the days and weeks to come. For simplicity, let’s write the ODE as \( x'' + 3x' + 2x = e^t \).

(1) For the homogeneous solutions, look for exponential solutions \( x = e^{rt} \) to the equation \( x'' + 3x' + 2x = 0 \). This gives \( r^2 e^{rt} + 3r e^{rt} + 2 e^{rt} = (r^2 + 3r + 2) e^{rt} = 0 \), so \( r^2 + 3r + 2 = (r+1)(r+2) = 0 \) \( \Rightarrow r = -1, r = -2 \). Individual homogeneous solutions are \( x_1(t) = e^{-t} \) and \( x_2(t) = e^{-2t} \), and by linearity any solution of the form \( x_h(t) = c_1 e^{-t} + c_2 e^{-2t} \) will satisfy the homogeneous ODE. It’s not hard to prove that these give all homogeneous solutions if we think of the 2nd order homogenous linear ODE as a composition of two 1st order linear ODEs and use the fact that we can always solve such equations. [See if you can complete the argument.]

Note that, in this case, the homogeneous solutions are transient.

(2) Once again, undetermined coefficients provide the simplest way to find a particular solution in this case. The obvious choice is to try a solution of the form \( x = Ae^t \). This gives \( x'' = A e^t \), and we get that \( A e^t \left( r^2 + 3r + 2 \right) = 0 \) \( \Rightarrow A = 1/6 \), so our particular solution is \( x_p(t) = 1/6 e^t \).

(3) Our general solution is then \( x(t) = c_1 e^{-t} + c_2 e^{-2t} + 1/6 e^t \). We compute \( x'(t) = -c_1 e^{-t} - 2c_2 e^{-2t} + 1/6 e^t \), and the initial conditions give
\[
\begin{align*}
  x(0) &= c_1 + c_2 + 1/6 = 4 \\
  x'(0) &= -c_1 - 2c_2 + 1/6 = 2 \\
\end{align*}
\] implies
\[
\begin{cases}
  c_1 + c_2 = 23/6 \\
  -c_1 - 2c_2 = 11/6 \\
\end{cases} \Rightarrow c_1 = 19/2, c_2 = -17/3.
\] So the unique solution to the initial value problem is \( x(t) = \frac{19}{2} e^{-t} - \frac{17}{3} e^{-2t} + \frac{1}{6} e^t \). 

Example #3 (sinusoidal input): Find the general solution to the ODE \( \frac{dx}{dt} + 2x = \cos 3t \)

Solution: As with all 1st order linear equations, solving using an integrating factor is always an option, though it could lead to some difficult integration. In this example, the integrating factor is \( e^{2t} \) which gives
\[
\frac{dx}{dt} + 2e^{2t}x = \frac{d}{dt}(e^{2t}x) = e^{2t} \cos 3t.
\] Integration gives \( e^{2t} x(t) = \int e^{2t} \cos 3t + C \) and \( x(t) = e^{-2t} \left[ \int e^{2t} \cos 3t + C \right] \).

The integration can be done using integration by parts (twice) and some additional algebra.

If we solve this using linearity:

(1) \( \frac{dx}{dt} + 2x = 0 \) gives the homogeneous solutions \( x_h(t) = ce^{-2t} \)

(2) For a particular solution, try \( x = a \cos 3t + b \sin 3t \). We calculate \( x' = 3b \cos 3t - 3a \sin 3t \), and substitution gives \( x' + 2x = (2a + 3b) \cos 3t + (3a - 2b) \sin 3t = \cos 3t \), so
\[
\begin{cases}
  2a + 3b = 1 \\
  -3a + 2b = 0 \\
\end{cases} \Rightarrow \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 2 \\ -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{2}{13} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{3}{13} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ so } x_p(t) = \frac{2}{13} \cos 3t + \frac{3}{13} \sin 3t \text{ or }
\]
\[
x_p(t) = \frac{1}{13}(2 \cos 3t + 3 \sin 3t).
\]

(3) The general solution is therefore \( x(t) = \frac{1}{13} e^{-2t} + \frac{2}{13} \cos 3t + \frac{3}{13} \sin 3t \).

A little trigonometry
Any expression of the form $a \cos \omega t + b \sin \omega t$ actually represents a single sinusoidal curve with frequency $\omega$ and an appropriate translation (phase shift), i.e., a function of the form $A \cos(\omega t - \phi_0)$. We can see this quickly using the sum of angle formula for cosine:

$$A \cos(\omega t - \phi_0) = A \cos \omega t \cos \phi_0 + A \sin \omega t \sin \phi_0 = a \cos \omega t + b \sin \omega t$$

We must therefore have

$$\begin{cases} A \cos \phi_0 = a \\ A \sin \phi_0 = b \end{cases}.$$ 

This is most easily understood with a right triangle as shown.

From this we see that $A = \sqrt{a^2 + b^2}$ and $\tan \phi_0 = \frac{b}{a}$.

In our example with $x_p(t) = \frac{1}{13} (2 \cos 3t + 3 \sin 3t)$ we would get $A = \frac{1}{13} \sqrt{2^2 + 3^2} = \frac{\sqrt{13}}{13} = \frac{1}{\sqrt{13}}$ and $\tan \phi_0 = \frac{3}{2}$. This gives $\phi_0 \approx 56.31^\circ$ or $\phi_0 \approx 0.9828$ radians. The period of the oscillation would be $\frac{2\pi}{\omega} = \frac{2\pi}{3}$.

Next time we’ll look at variation of parameters as another tool for finding particular solutions, and we’ll introduce complex numbers and related facts to reformulate some of the methods involving exponential and sinusoidal inputs.

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