Concourse 18.03 – Lecture #3

Today's lecture builds on the idea of linearity in the context of ordinary differential equations and how this can be used to produce all solutions to an *n*th order linear ordinary differential equation. Recall the basic definition and method from the previous lecture:

Definition: A differential equation of the form $\frac{d^n x}{dt^n} + p_{n-1}(t) \frac{d^{n-1}x}{dt^{n-1}} + \dots + p_1(t) \frac{dx}{dt} + p_0(t)x(t) = q(t)$, where $p_{n-1}(t), \dots, p_1(t), p_0(t), q(t)$ are functions of the independent variable *t*, is called an *n*th order linear ordinary differential equation. In the case where q(t) = 0 for all *t*, we call the equation homogeneous. Otherwise we call it inhomogeneous.

Linearity method using homogeneous solutions and particular solutions

Suppose we have an inhomogeneous linear ODE of the form T(f) = g where *T* is an *n*th order linear differential operator. We can produce ALL solutions to T(f) = g as follows:

- (1) First solve the homogeneous equation T(f) = 0 to find a general expression for all such solutions. Call this the homogeneous solution f_h . It will generally involve *n* arbitrary constants.
- (2) Find a single particular solution to the inhomogeneous equation T(f) = g. Call this particular solution f_p .
- (3) The general solution to T(f) = g is then $f = f_h + f_p$.

Analogy with systems of linear equations

Suppose we want to solve a consistent, inhomogeneous system of linear algebraic equations. In matrix form, if the system is represented as $A\mathbf{x} = \mathbf{b}$ where \mathbf{A} is an $m \times n$ matrix, and if \mathbf{x}_h represents all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ and \mathbf{x}_p is a single solution to $A\mathbf{x} = \mathbf{b}$, then all solutions to $A\mathbf{x} = \mathbf{b}$ will be of the form $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$. Typically, these homogeneous solutions are lines, planes or higher-dimensional analogues (subspaces) passing through the origin. This just says that the inhomogeneous solutions are parallel translates of these subspaces.

Example: Find all solutions of the linear system $\begin{cases} x & -2z = 3\\ 2x - y - 5z = 2\\ 3x - y - 7z = 5 \end{cases}$. We can solve this most easily by row

reduction to get an equivalent system from which we can readily express all solutions. Specifically, we have:

[1	0	-2	3]		1	0	-2	3		[1	0	-2	3		$\left(x-2z=3\right)$		$\int x = 3 + 2t$		$\begin{bmatrix} x \end{bmatrix}$		[3]	[2
2	-1	-5	2	\rightarrow	0	1	1	4	\rightarrow	0	1	1	4	\Rightarrow	$\begin{cases} y+z=4 \end{cases}$	\Rightarrow	$\begin{cases} y = 4 - t \end{cases}$	$\rangle \Rightarrow$	y	=	4	+t	-1
3	-1	-7	5		0	1	1	4		0	0	0	0		z arbitrary		z = t	J	$\lfloor z \rfloor$		0		1

If we were to solve the corresponding homogeneous linear system $\begin{cases} x & -2z = 0 \\ 2x - y - 5z = 0 \\ 3x - y - 7z = 0 \end{cases}$, the process is similar:

$$\begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 2 & -1 & -5 & | & 0 \\ 3 & -1 & -7 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \begin{cases} x - 2z = 0 \\ y + z = 0 \\ z \text{ arbitrary} \end{cases} \Rightarrow \begin{cases} x = 2t \\ y = t \\ z = t \end{cases} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

The only difference is that the inhomogeneous solutions differ from the homogeneous solutions by a particular solution (which corresponds to t = 0).

Back to solving differential equations

Example #1: Solve the initial value problem $\frac{dy}{dx} + xy = 2x$; y(0) = 5.

(1) First, we solve the homogeneous equation $\frac{dy}{dx} + xy = 0$. This will always be separable. We get $\frac{dy}{dx} = -xy$ and $\frac{dy}{y} = -xdx$, so $\int \frac{dy}{y} = -\int xdx \implies \ln|y| = -\frac{1}{2}x^2 + C \implies y_h = Ae^{-\frac{1}{2}x^2}$.

(2) Next, we seek a particular solution to $\frac{dy}{dx} + xy = 2x$. The Method of Undetermined Coefficients is a good choice here based on the relatively simple functions involved. If we try a solution of the form $y_p = ax^2 + bx + c$ (which is actually more general that we really need), we have $\frac{dy}{dx} = 2ax + b$, so substitution into the ODE gives:

$$(2ax+b) + x(ax2 + bx + c) = ax3 + bx2 + (2a+c)x + b = 2x$$

So we must have $a = 0, b = 0, 2a + c = 2, b = 0 \implies a = 0, b = 0, c = 2 \implies y_p = 2$. (3) So the general solution must be $y = y_p + y_p = 4e^{-\frac{1}{2}x^2} + 2$. The initial value gives $x = 1 + 2e^{-\frac{1}{2}x^2} + 2e^{-\frac{1}{$

(3) So, the general solution must be $y = y_h + y_p = Ae^{-\frac{1}{2}x^2} + 2$. The initial value gives y(0) = A + 2 = 5, so A = 3 and the unique solution to the initial value problem is $y = 3e^{-\frac{1}{2}x^2} + 2$.

<u>Note</u>: This problem could also have been solved using the integrating factor $e^{\frac{1}{2}x^2}$ derived by the method already discussed. This would give $e^{\frac{1}{2}x^2} \frac{dy}{dx} + xe^{\frac{1}{2}x^2}y = \frac{d}{dx}\left(e^{\frac{1}{2}x^2}y\right) = 2xe^{\frac{1}{2}x^2} \implies e^{\frac{1}{2}x^2}y = 2e^{\frac{1}{2}x^2} + C \implies y = 2 + Ce^{-\frac{1}{2}x^2}$, as above.

Example #2: Solve the initial value problem $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin x$, y(0) = 1, y'(0) = 2.

This problem cannot be done using an integrating factor as that's really a method specific to 1st order linear equations. So we proceed using our methods based on linearity.

(1) First we seek <u>homogeneous solutions</u>, i.e. solutions of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$. We're getting a little ahead of ourselves here, but for a linear ODE with constant coefficients we begin by seeking exponential solutions of the form $y = e^{rx}$. The logic behind this choice will be developed soon, but differentiation gives $\frac{dy}{dx} = re^{rx}$ and $\frac{d^2y}{dx^2} = r^2e^{rx}$. Substitution into the ODE gives $r^2e^{rx} - 3re^{rx} + 2e^{rx} = (r^2 - 3r + 2)e^{rx} = 0$. This can only vanish when $r^2 - 3r + 2 = (r-1)(r-2) = 0$, so either r = 1 or r = 2. Therefore $y_1 = e^x$ and $y_2 = e^{2x}$ are solutions. Now here's where linearity becomes especially useful. If T(y) = 0 is the form of the homogeneous equation (so

T(y_1) = 0 and $T(y_2)$ = 0), then any function of the form $c_1y_1 + c_2y_2$ will also satisfy the homogeneous equation, i.e. $T(c_1y_1 + c_2y_2) = c_1T(y_1) + c_2T(y_2) = c_1 \cdot 0 + c_2 \cdot 0 = 0$. So $y_h = c_1e^x + c_2e^{2x}$ will give homogeneous solutions for any scalars c_1, c_2 . Though we have not yet shown it, the fact is that these give <u>all</u> of the homogeneous solutions.

(2) Now let's concentrate on getting a <u>particular solution</u> to the original inhomogeneous equation. If you think about what kinds of functions might be such that when combined with its 1st and 2nd derivatives in the manner prescribed by the ODE to yield the function $\sin x$, it should be pretty clear that something of the form

 $y_{p} = A\sin x + B\cos x \text{ is a likely candidate. We have } \begin{cases} y_{p} = A\sin x + B\cos x \\ y'_{p} = -B\sin x + A\cos x \\ y''_{p} = -A\sin x - B\cos x \end{cases}, \text{ so:}$

$$y_p'' - 3y_p' + 2y_p = (-A + 3B + 2A)\sin x + (-B - 3A + 2B)\cos x = (A + 3B)\sin x + (-3A + B)\cos x = \sin x$$

This implies that
$$\begin{cases} A+3B=1\\ -3A+B=0 \end{cases} \implies \begin{bmatrix} 1 & 3\\ -3 & 1 \end{bmatrix} \begin{bmatrix} A\\ B \end{bmatrix} = \begin{bmatrix} 1\\ 0 \end{bmatrix} \implies \begin{bmatrix} A\\ B \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 1 & -3\\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1\\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{10}\\ \frac{3}{10} \end{bmatrix} \implies \begin{cases} A=\frac{1}{10}\\ B=\frac{3}{10} \end{cases}.$$

So $y_p = \frac{1}{10} \sin x + \frac{3}{10} \cos x$ is a particular solution.

(3) Therefore all solutions are of the form $y(x) = c_1 e^x + c_2 e^{2x} + \frac{1}{10} \sin x + \frac{3}{10} \cos x$

Finally, to solve the given initial value problem, note that $y'(x) = c_1 e^x + 2c_2 e^{2x} + \frac{1}{10} \cos x - \frac{3}{10} \sin x$, so:

$$\begin{cases} y(0) = c_1 + c_2 + \frac{3}{10} = 1\\ y'(0) = c_1 + 2c_2 + \frac{1}{10} = 2 \end{cases} \implies \begin{cases} c_1 + c_2 = \frac{7}{10}\\ c_1 + 2c_2 = \frac{19}{10} \end{cases} \implies c_2 = \frac{6}{5}, c_1 = -\frac{1}{2} \implies y(x) = -\frac{1}{2}e^x + \frac{6}{5}e^{2x} + \frac{1}{10}\sin x + \frac{3}{10}\cos x \end{cases}$$

The next topic we'll discuss is the Input-Response formalism for understanding *n*th order linear inhomogeneous ordinary differential equations (if you don't mind all the adjectives).

Notes by Robert Winters