Concourse 18.03 – Lecture #16

In today's lecture we continue the discussion of the Laplace Transform complete with several more calculations We'll also solve some Initial Value Problems by applying the Laplace Transform directly. As we prepare for Convolution (sounds like a Great Awakening of sorts) we'll also review the idea of ZSR + ZIR, i.e. constructing a solution to an Initial Value Problem by summing the Zero State Response and the Zero Input Response, a.k.a. Two Wrongs Make a Right (which is, of course, only true in this very isolated situation).

Definition (again): The ***Laplace transform*** of a function $f(t)$ is defined by $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} e^{-st} f(t)dt$ where the new (complex) variable s is such that its real part $\mathrm{Re}(s) \gg 0$ (the integral would otherwise not converge). Note that the lower limit of the integral indicates that $t=0$ is included and is intended to address potential discontinuities and delta functions. We use the convention that a function of t will be represented by a lower case name and its Laplace transform by the corresponding upper case name, e.g. $\mathcal{L}[x(t)] = X(s)$.

Some More Calculations (continuing where we left off)

11) ***t*-shift rule**: $\mathcal{L}[f(t-a)] = e^{-as}F(s)$ if $a \ge 0$ and $f(t) = 0$ for $t < 0$.

This may also be expressed as $\mathcal{L}[f_a(t)] = e^{-as}F(s)$ where $f_a(t) = u(t-a)f(t-a) = \begin{Bmatrix} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{Bmatrix}$.

Starting with the definition $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} e^{-st} f(t)dt$, we make the substitution $u = t-a$, $du = dt$ and change the limits on the integral to get:

$$\mathcal{L}[f(t-a)] = \int_{0-}^{\infty} e^{-st} f(t-a)dt = \int_{a-}^{\infty} e^{-st} f(t-a)dt = \int_{0-}^{\infty} e^{-s(u+a)} f(u)du$$

$$= \int_{0-}^{\infty} e^{-as}e^{-su} f(u)du = e^{-as}\int_{0-}^{\infty} e^{-su} f(u)du = e^{-as}F(s)$$

12) $\mathcal{L}[\delta(t-a)] = \mathcal{L}[\delta_a(t)] = e^{-as}$

We calculate $\mathcal{L}[\delta(t-a)] = \int_{0-}^{\infty} e^{-st}\delta(t-a)dt = e^{-st}\Big|_{t=a} = e^{-as}$

This also follows immediately from the *t*-shift rule and the fact that $\mathcal{L}[\delta(t)] = 1$.

13) $\mathcal{L}[u(t-a)] = \mathcal{L}[u_a(t)] = \dfrac{e^{-as}}{s}$

This follows immediately from the *t*-shift rule and the fact that $\mathcal{L}[1] = \mathcal{L}[u(t)] = \dfrac{1}{s}$.

14) $\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$

This follows by repeated application of the *s*-derivative rule: $\mathcal{L}[t f(t)] = -F'(s)$.

15) $\boxed{\mathcal{L}[e^{at}\cos(\omega t)] = \dfrac{s-a}{(s-a)^2+\omega^2}}$ and $\boxed{\mathcal{L}[e^{at}\sin(\omega t)] = \dfrac{\omega}{(s-a)^2+\omega^2}}$

These follow immediately from the **s-shift rule**, $\mathcal{L}[e^{rt}f(t)] = F(s-r)$ together with the transforms for the cosine and sine functions, i.e. $\mathcal{L}[\cos(\omega t)] = \dfrac{s}{s^2+\omega^2}$ and $\mathcal{L}[\sin(\omega t)] = \dfrac{\omega}{s^2+\omega^2}$.

16) $\mathcal{L}[t\cos(\omega t)] = \dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$ and $\mathcal{L}[t\sin(\omega t)] = \dfrac{2\omega s}{(s^2+\omega^2)^2}$

These both follow immediately from the *s*-derivative rule: $\mathcal{L}[t f(t)] = -F'(s)$ by differentiating the transforms of the sine and cosine functions, i.e. $\mathcal{L}[\cos(\omega t)] = \dfrac{s}{s^2+\omega^2}$ and $\mathcal{L}[\sin(\omega t)] = \dfrac{\omega}{s^2+\omega^2}$.

We'll add to this list after we introduce ***convolution*** in the next lecture.

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Unit Impulse Response

Unit impulse response refers to the solution of the ODE $p(D)[x(t)] = \delta(t)$ with rest initial conditions. The solution is also known as the ***weight function*** for the given differential operator $p(D)$. It is the simplest to tackle algebraically and we'll use it soon along with convolution to solve Initial Value Problems. We generally denote the unit impulse response (weight function) by $w(t)$. It's Laplace Transform $W(s)$ is called the ***transfer function***.

Unit Step Response

Unit step response refers to the solution of the ODE $p(D)[x(t)] = u(t)$ with rest initial conditions. It is a bit more algebraically complicated to solve than the unit impulse response but is still relatively simple. We generally denote the unit step response by $v(t)$.

It is worth noting that because these differential operators are time-invariant (constant coefficients), we can use the generalized derivative to differentiate both sides of $p(D)[x(t)] = u(t)$ to get $D \circ p(D)[v(t)] = p(D)[D(v(t))] = p(D)[\dot{v}(t)] = D[u(t)] = \delta(t)$, so $p(D)[\dot{v}(t)] = \delta(t)$. Therefore $\dot{v}(t) = w(t)$.

Example 1: Find the unit impulse response and the unit step response for the operator $p(D) = D + 3I$.

Solution: For the unit impulse response we solve $\dot{w} + 3w = \delta(t)$ with rest initial conditions. Transforming both sides gives $p(s)W(s) = (s+3)W(s) = 1$, so $W(s) = \dfrac{1}{p(s)} = \dfrac{1}{s+3}$. This is just $\mathcal{L}(e^{-3t})$, so $\boxed{w(t) = e^{-3t}}$.

For the unit step response we solve $\dot{v} + 3v = u(t)$ with rest initial conditions. Transforming both sides gives $p(s)V(s) = (s+3)V(s) = \dfrac{1}{s}$, so $V(s) = \dfrac{1}{s(s+3)} = \tfrac{1}{3}\left(\dfrac{1}{s} - \dfrac{1}{s+3}\right)$. It follows that $\boxed{v(t) = \tfrac{1}{3}(1 - e^{-3t})}$.

Example 2: Find the unit impulse response for the operator $p(D) = D^2 + \omega^2$ where ω is a given positive constant (natural frequency for a harmonic oscillator).

Solution: For the unit impulse response we solve $\ddot{w} + \omega^2 w = \delta(t)$ with rest initial conditions. Transforming both sides gives $p(s)W(s) = (s^2 + \omega^2)W(s) = 1$, so $W(s) = \dfrac{1}{p(s)} = \dfrac{1}{s^2 + \omega^2}$. Adjusting the coefficients to write this as $W(s) = \dfrac{1}{\omega}\left(\dfrac{\omega}{s^2 + \omega^2}\right)$ we deduce from our table of transforms that $\boxed{w(t) = \tfrac{1}{\omega}\sin(\omega t)}$.

ZIR + ZSR

Given an n-th order linear ODE $p(D)[x(t)] = f(t)$ with initial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = \dot{x}_0$, …, $x^{(n-1)}(t_0) = x_0^{(n-1)}$, we refer to the case where $x(t_0) = 0$ and $\dot{x}(t_0) = 0$, …, $x^{(n-1)}(t_0) = 0$ as the **zero state**. If we solve $p(D)[x(t)] = f(t)$ for the zero state, we refer to this solution $x_{ZSR}(t)$ as the **zero state response (ZSR)**.

If we seek homogeneous solutions to the ODE $p(D)[x(t)] = 0$ with initial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = \dot{x}_0$, …, $x^{(n-1)}(t_0) = x_0^{(n-1)}$, this will have a unique solution $x_{ZIR}(t)$ called the **zero input response (ZIR)**.

The general solution to the ODE $p(D)[x(t)] = f(t)$ will be $x(t) = x_h(t) + x_p(t)$ for some particular solution $x_p(t)$ and homogeneous solutions $x_h(t)$, and we would then use the initial conditions to determine any unknown coefficients. However, note that the zero state response (ZSR) is a particular solution and the zero input response is a (single) homogeneous solution. If we let $x(t) = x_{ZIR}(t) + x_{ZSR}(t)$, note that:

$$\left\{\begin{array}{c} x(t_0) = x_{ZIR}(t_0) + x_{ZSR}(t_0) = x_{ZIR}(t_0) + 0 = x_{ZIR}(t_0) = x_0 \\ \dot{x}(t_0) = \dot{x}_{ZIR}(t_0) + \dot{x}_{ZSR}(t_0) = \dot{x}_{ZIR}(t_0) + 0 = \dot{x}_{ZIR}(t_0) = \dot{x}_0 \\ \vdots \\ x^{(n-1)}(t_0) = x_{ZIR}{}^{(n-1)}(t_0) + x_{ZSR}{}^{(n-1)}(t_0) = x_{ZIR}{}^{(n-1)}(t_0) + 0 = x_{ZIR}{}^{(n-1)}(t_0) = x_0{}^{(n-1)} \end{array}\right\}$$

so $x(t) = x_h(t) + x_p(t)$ satisfies the initial value problem (IVP) without the need to introduce any additional constants. That is, $x(t) = \mathbf{ZIR} + \mathbf{ZSR}$.

This observation is very helpful when solving initial value problems using Laplace Transform methods – specifically when we use the Unit Impulse Response together with ***convolution*** to solve for the zero state response (ZSR). More on that later.

Example 3: Solve the IVP $\frac{dx}{dt} + 3x = 3\cos 2t$ with initial value $x(0-) = 2$ (the $0-$ is just for emphasis here).

Solution: First, it should be emphasized that for a problem like this our previous methods work well and there is no particular need to use Laplace transform methods. That said, we proceed with two different approaches.

Laplace Direct: For this we simply transform both sides of the equation mindful of the need to incorporate the initial condition as we transform the derivative. This gives:

$$sX(s) - 2 + 3X(s) = (s+3)X(s) - 2 = \frac{3s}{s^2+4}, \text{ so } (s+3)X(s) = 2 + \frac{3s}{s^2+4} = \frac{2s^2+3s+8}{s^2+4}.$$

Therefore $X(s) = \dfrac{2s^2+3s+8}{(s+3)(s^2+4)} = \dfrac{A}{s+3} + \dfrac{Bs+C}{s^2+4}$.

Clearing fractions gives $2s^2 + 3s + 8 = A(s^2+4) + (s+3)(Bs+C)$

There are several good ways to proceed. First, if we choose convenient points we might first choose $s = -3$ to quickly conclude that $17 = 13A$, so $\boxed{A = \frac{17}{13}}$. You might think the well has run dry, but we are free to use complex numbers. If we choose $s = 2i$ (and as we'll see we won't even have to separately consider its complex conjugate) we get $-8 + 6i + 8 = 6i = (3+2i)(2Bi + C) = (-4B + 3C) + i(6B + 2C)$. We can equate both real and imaginary parts to conclude that $-4B + 3C = 0$ and $6B + 2C = 6$. These give $\boxed{B = \frac{9}{13}}$ and $\boxed{C = \frac{12}{13}}$.

Thus $\boxed{X(s) = \frac{17}{13}\left(\frac{1}{s+3}\right) + \frac{9}{13}\left(\frac{s}{s^2+4}\right) + \frac{6}{13}\left(\frac{2}{s^2+4}\right)}$. So $\boxed{x(t) = \frac{17}{13}e^{-3t} + \frac{9}{13}\cos 2t + \frac{6}{13}\sin 2t}$.

Alternatively, we could simply multiply out and collect terms to get $2s^2 + 3s + 8 = (A+B)s^2 + (3B+C)s + (4A+3C)$ and then use your favorite linear algebra method to derive the same results as above.

ZSR+ZIR (not really recommended here): If we first solve $\frac{dx}{dt} + 3x = 3\cos 2t$ with rest initial conditions we get $(s+3)X(s) = \dfrac{3s}{s^2+4}$ and $X(s) = \dfrac{3s}{(s+3)(s^2+4)} = -\frac{9}{13}\left(\dfrac{1}{s+3}\right) + \frac{9}{13}\left(\dfrac{s}{s^2+4}\right) + \frac{6}{13}\left(\dfrac{2}{s^2+4}\right)$. So $\boxed{x_{ZSR}(t) = -\frac{9}{13}e^{-3t} + \frac{9}{13}\cos 2t + \frac{6}{13}\sin 2t}$. Next we seek the zero input response, so we solve $\frac{dx}{dt} + 3x = 0$ with $x(0) = 2$. This quickly gives $\boxed{x_{ZIR}(t) = 2e^{-3t}}$. Combining these gives $\boxed{x(t) = \frac{17}{13}e^{-3t} + \frac{9}{13}\cos 2t + \frac{6}{13}\sin 2t}$.

We'll soon do this a third way – via ***convolution***.

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} e^{-st} f(t)dt \quad \text{for } \operatorname{Re}(s) \gg 0$.

1. Linearity: $\mathcal{L}[af(t)+bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] = aF(s)+bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{rt} f(t)] = F(s-r)$.

4. t-shift rule: $\mathcal{L}[f(t-a)] = e^{-as}F(s) \quad \text{if } a \ge 0 \text{ and } f(t)=0 \text{ for } t<0$.

5. s-derivative rule: $\mathcal{L}[t\,f(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s) - f(0-) \qquad \mathcal{L}[f''(t)] = s^2F(s) - sf(0-) - f'(0-)$

$$\mathcal{L}[f^{(n)}(t)] = s^nF(s) - s^{n-1}f(0-) - s^{n-2}f'(0-) - \cdots - f^{(n-1)}(0-)$$

7. Convolution rule: $\mathcal{L}[f(t)*g(t)] = F(s)G(s)$, $\quad (f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s)$, $w(t)$ the unit impulse response.

   If $q(t)$ is regarded as the input signal in $p(D)x = q(t)$, $W(s) = \frac{1}{p(s)}$.

Formulas for the Laplace transform

$\mathcal{L}[1] = \dfrac{1}{s}$

$\mathcal{L}[\delta(t)] = 1$

$\mathcal{L}[\delta(t-a)] = \mathcal{L}[\delta_a(t)] = e^{-as}$

$\mathcal{L}[u(t-a)] = \mathcal{L}[u_a(t)] = \dfrac{e^{-as}}{s}$

$\mathcal{L}[e^{at}] = \dfrac{1}{s-a}$

$\mathcal{L}[t] = \dfrac{1}{s^2}$

$\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$

$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$

$\mathcal{L}[u(t-a)f(t-a)] = e^{-as}F(s)$

$\mathcal{L}[u(t-a)f(t)] = e^{-as}\mathcal{L}[f(t+a)]$

$\mathcal{L}[\cos(\omega t)] = \dfrac{s}{s^2+\omega^2}$

$\mathcal{L}[\sin(\omega t)] = \dfrac{\omega}{s^2+\omega^2}$

$\mathcal{L}[t\cos(\omega t)] = \dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$

$\mathcal{L}[t\sin(\omega t)] = \dfrac{2\omega s}{(s^2+\omega^2)^2}$

$\mathcal{L}[e^{zt}\cos(\omega t)] = \dfrac{s-z}{(s-z)^2+\omega^2}$

$\mathcal{L}[e^{zt}\sin(\omega t)] = \dfrac{\omega}{(s-z)^2+\omega^2}$

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