Concourse 18.03 – Lecture #15

In today's lecture we begin to address the situation of linear *n*th order ODEs with **discontinuous and/or non-differentiable inputs**. The method we'll develop (**Laplace Transform**) will be applicable to other types of inputs, but it's especially relevant when dealing with discontinuous inputs and inputs defined only by numerical data.

The Main Idea: Beginning with a **linear *n*th order ODE with initial conditions** (an initial value problem), we'll transform this into an **algebraic equation**, solve this equation, and then **transform back** in order to produce a solution to the initial value problem. We are only concerned with the solution for $t > 0$.

Big Idea #1: Generalized functions, a.k.a. a function is only as good as how it is integrated - in particular, delta functions and step functions.

Big Idea #2: We'll devise a systematic way of formally solving an ODE with such inputs, and then use integration (convolution) to produce solutions to any given initial value problem.

Suppose that $g(t)$ is a function with ***compact support***, i.e. it vanishes outside some closed and bounded interval. We would like to consider two functions $f_1(t)$ and $f_2(t)$ to be **equivalent *in the sense of measurement*** if for all functions $g(t)$ with compact support, they integrate in the same way, i.e. $\int_{-\infty}^{+\infty} f_1(t)g(t)dt = \int_{-\infty}^{+\infty} f_2(t)g(t)dt$. Said differently, $\int_{-\infty}^{+\infty} [f_1(t) - f_2(t)]g(t)dt = 0$ for all functions $g(t)$ with compact support. It's not hard to see that for continuous functions this means that necessarily $f_1(t) = f_2(t)$ for all t, but we're really interested in what this means for *discontinuous* functions and functions with *impulses*, i.e. "delta functions".

Heaviside functions, box functions, and delta functions

The Heaviside function [named for Oliver Heaviside (1850–1925)] is $u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 1 \end{cases}$. For our purposes it really doesn't matter how it is defined at $t = 0$, because it's not relevant when integrating this function. We can also define translated Heaviside functions $u_a(t) = u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$. These functions can be scaled and added to represent functions corresponding to "switching on and off". For example, we can represent the function $f(t) = \begin{cases} 0 & t < 3 \\ 4 & 3 < t < 5 \\ 0 & t > 5 \end{cases} = 4[u_3(t) - u_5(t)]$. This is called a **box function**. We can combine box functions as necessary, e.g. $g(t) = \begin{cases} 0 & t < 3 \\ 4 & 3 < t < 5 \\ 1 & 5 < t < 6 \\ 0 & t > 6 \end{cases} = 4[u_3(t) - u_5(t)] + 1[u_5(t) - u_6(t)] = 4u_3(t) - 3u_5(t) - u_6(t)$.

The Heaviside function is constant everywhere except at $t = 0$, and because it has a jump discontinuity there we usually just say that it's not differentiable at $t = 0$. However, we could heuristically observe that by considering points immediately to the left and right of the discontinuity any continuous approximation to this function would have to have a very large slope in the vicinity of $t = 0$. We might at least try to express this by saying that $u'(t) = \begin{cases} 0 & t < 0 \\ \infty & t = 0 \\ 0 & t > 1 \end{cases} = \delta(t)$, the so-called delta function, but this doesn't really make much sense in terms of traditional functions. We may, however, still be able to make sense out of this if we take the view that *a function is only as good as how it is integrated*. Similarly, $\dot{u}(t-a) = \delta(t-a)$, a translated delta function.

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Digression – Linear functionals and measurement

One of the most common things we do in vector calculus is finding the component or scalar projection of a vector in $\mathbf{R}^n$ in a given direction. The tool used to accomplish this task is the dot product. If $\mathbf{u}$ is a unit vector we have that $\text{comp}_{\mathbf{u}}(\mathbf{v}) = \mathbf{v}\cdot\mathbf{u}$. This is, in fact, a linear function from $\mathbf{R}^n$ to $\mathbf{R}$, i.e. $\mathbf{v}\in\mathbf{R}^n \;\to\; \mathbf{v}\cdot\mathbf{u}\in\mathbf{R}$. These are called **linear functional**s. Indeed, the standard components of a vector $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ in $\mathbf{R}^3$ are "measured" by noting that $v_1 = \mathbf{v}\cdot\mathbf{i}$, $v_2 = \mathbf{v}\cdot\mathbf{j}$, and $v_3 = \mathbf{v}\cdot\mathbf{k}$ using the standard unit vectors as a basis for $\mathbf{R}^3$.

If we let $L(\mathbf{v}) = \text{comp}_{\mathbf{u}}(\mathbf{v}) = \mathbf{v}\cdot\mathbf{u}$, we see that
$L(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = (c_1\mathbf{v}_1 + c_2\mathbf{v}_2)\cdot\mathbf{u} = c_1\mathbf{v}_1\cdot\mathbf{u} + c_2\mathbf{v}_2\cdot\mathbf{u} = c_1L(\mathbf{v}_1) + c_2L(\mathbf{v}_2)$, so L is linear.

The Fourier coefficients are just the "measure" of how much of a given periodic function is associated with each "mode". It's really no different that calculating the component of a vector in specific directions.

$$a_0 = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt \qquad a_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \qquad b_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

In this case, each of the calculations of Fourier coefficients takes a (periodic) function and produces a real number, e.g. $a_n = L(f) = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$.

Note that just as was the case with vectors and dot products, L acts linearly, i.e.

$$L(c_1f_1 + c_2f_2) = \tfrac{1}{\pi}\int_{-\pi}^{\pi}[c_1f_1(t) + c_2f_2(t)]\cos nt\,dt = c_1\left(\tfrac{1}{\pi}\int_{-\pi}^{\pi} f_1(t)\cos nt\,dt\right) + c_2\left(\tfrac{1}{\pi}\int_{-\pi}^{\pi} f_2(t)\cos nt\,dt\right)$$
$$= c_1L(f_1) + c_2L(f_2)$$

So ***L* is also a linear functional**, though in this case *it takes functions and produces real numbers*.

There is, however, one linear functional, arguably the simplest imaginable one, that we don't usually think of in terms of integration (though maybe we should), namely ***evaluation***. Specifically, if $f(t)$ is a function, we can, for any specific value $t = a$, consider $L_a(f) = f(a)$. It's quite simple to see that
$L_a(c_1f_1 + c_2f_2) = (c_1f_1 + c_2f_2)(a) = c_1f_1(a) + c_2f_2(a) = c_1L_a(f_1) + c_2L_a(f_2)$, so L_a is a linear functional. It is not defined in terms of integration, but we will find it useful to do so nonetheless.

Generalized functions

You can heuristically think of the step function $u(t)$ as any nice smooth function which is 0 for $t < -\varepsilon$ and 1 for $t > \varepsilon$, where ε is a positive number which is much smaller than any time scale we care about in the context we are studying at the moment. Similarly, the best way for you to understand the "delta function" (defined below) is to think of it as any smooth function which is zero except in the immediate neighborhood of $t = 0$ and which has integral 1. As we'll see, we can also think of the delta functions $\delta(t)$ and $\delta(t-a)$ as the "function you integrate against" in order to evaluate a function at respectively $t = 0$ and at any $t = a$. That is, $\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$ and $\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$. How can we make sense of this?

Making the most of integration by parts

In first-year Calculus we learned that for differentiable functions $u(t)$ and $v(t)$ the Product Rule applies, i.e. $\frac{d}{dt}[u(t)v(t)] = u(t)v'(t) + v(t)u'(t)$.

On any finite interval $[a,b]$ we can integrate both sides of the Product Rule this and apply the Fundamental Theorem of Calculus to get that $u(b)v(b) - u(a)v(a) = \int_a^b \frac{d}{dt}[u(t)v(t)]\,dt = \int_a^b u(t)v'(t)dt + \int_a^b v(t)u'(t)dt$. Though we often think of Integration by Parts as defined formally by $\int u dv = uv - \int v du$, the stated result is really what this means, i.e. we can say that $\int_a^b u(t)v'(t)dt = [uv]_a^b - \int_a^b v(t)u'(t)dt$.

If one of these functions has compact support, i.e. if it vanishes outside of some close, bounded interval, then we can extend the result to the entire real line and simplify the statement considerably (since the value of the product of the functions will vanish outside some interval. Specifically, if $g(t)$ has compact support and if $f(t)$ is any function, we can say that $\boxed{\int_{-\infty}^{+\infty} f'(t)g(t)dt = -\int_{-\infty}^{+\infty} f(t)g'(t)dt}$. We can actually use this to define a derivative $f'(t)$ in a generalized way, i.e. a ***generalized derivative***. It rests on the notion that functions can be understood by *how they are integrated* and not just by *how they are evaluated*. These ***generalized functions*** are also known as ***distributions***.

Perhaps the most important illustration of this is the generalized derivative of the Heaviside function $u(t)$. We formally called this the delta function $u'(t) = \delta(t)$ even though it didn't really make sense at the point of discontinuity $t = 0$ for the Heaviside function. However, we can say that if $u'(t) = \delta(t)$ then for any function $g(t)$ with compact support:

$$\int_{-\infty}^{+\infty} \delta(t)g(t)dt = \int_{-\infty}^{+\infty} u'(t)g(t)dt = -\int_{-\infty}^{+\infty} u(t)g'(t)dt = -\int_{0}^{+\infty} g'(t)dt = -[0 - g(0)] = g(0) .$$

That is, if we "integrate a function against the delta function", this is simply evaluation of that function at 0. It is this really this property that defines the delta function as a generalized function.

Similarly, we can do the same for the translated Heaviside function $u_a(t) = u(t-a)$ to conclude that its generalized derivative $\delta_a(t) = \delta(t-a)$ is such that for any function $g(t)$ with compact support:

$$\int_{-\infty}^{+\infty} g(t)\delta_a(t)dt = \int_{-\infty}^{+\infty} g(t)\delta(t-a)dt = g(a) .$$

You can also take a sequential approach to make sense of this in terms of limits, i.e. if you successively approximate the delta function by a sequence of continuous functions $f_k(t)$ where the support (domain where it's nonzero) gets narrower $[-\varepsilon_k, +\varepsilon_k]$ and the values grow reciprocally in such a way that at each step the integral is always $\int_{-\infty}^{+\infty} f_k(t)dt = \int_{-\varepsilon_k}^{+\varepsilon_k} f_k(t)dt = 1$ (we call such functions probability densities), then you can show that $\lim_{k\to\infty}\left[\int_{-\infty}^{+\infty} g(t) f_k(t)dt\right] = g(0)$.

Note: The Fundamental Theorem of Calculus as well as all the usual rules of differentiation also apply to generalized derivatives, so we actually have a "generalized calculus" for dealing with these generalized functions or distributions (though it may take a while getting used to it). Basically, we extend the usual rules of differentiation to generalized functions and together with the fact that $\dot{u}(t-a) = \delta(t-a)$.

A function $f(t)$ is "regular" or "piecewise smooth" if it can be broken into pieces each having all higher derivatives and such that at each breakpoint $f^{(n)}(a-)$ and $f^{(n)}(a+)$ exist. A "singularity function" is a linear combination of shifted delta functions. A "**generalized function**" $f(t)$ is a sum $f(t) = f_r(t) + f_s(t)$ of a regular function and a singularity function. Any regular function $f(t)$ has a "**generalized derivative**" $f'(t)$, with regular part $f_r'(t)$ the regular derivative of $f(t)$ wherever it exists, and singular part $f_s'(t)$ given by a sum of terms $(f(a+) - f(a-))\delta(t-a)$ as a runs over the discontinuities of f.

Now, to get back to the Main Idea, how can we solve a linear differential equation $[p(D)]x(t) = q(t)$ by transforming it into an algebraic equation, solving that algebraic equations, and then transforming back to produce a solution to an initial value problem? As we will only be concerned with forward time, we'll presume that $q(t)$ satisfies $q(t) = 0$ for $t < 0$.

The Laplace Transform

Definition: The ***Laplace transform*** of a function $f(t)$ is defined by $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} e^{-st} f(t)dt$ where the new (complex) variable s is such that its real part $\text{Re}(s) \gg 0$ (the integral would otherwise not converge). Note that the lower limit of the integral indicates that $t = 0$ is included and is intended to address potential discontinuities and delta functions.

We will liberally make use of the convention that a function of t will be represented by a lower case name and its Laplace transform by the corresponding upper case name, e.g. $\mathcal{L}[x(t)] = X(s)$.

Linearity
Because the Laplace transform is defined as an integral, it's easy to see that
$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] = aF(s) + bG(s)$.

Specifically,

$$\mathcal{L}[af(t) + bg(t)] = \int_{0-}^{\infty} e^{-st}[af(t) + bg(t)]dt = a\int_{0-}^{\infty} e^{-st} f(t)dt + b\int_{0-}^{\infty} e^{-st} g(t)dt = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] = aF(s) + bG(s)$$

This will permit us to transform a differential equation term-by-term (and transform back as well).

Inverse transform: $F(s)$ essentially determines $f(t)$ for $t \ge 0$. This will generally allow us to produce solutions to a given Initial Value Problem by simply recognizing, term by term, a solution by identifying which functions gave rise to each term of the transformed differential equation.

Some Calculations

1) For our purposes, since we are only concerned with $t \ge 0$, the constant function $f(t) = 1$ and the Heaviside function $u(t) = \begin{Bmatrix} 0 & t < 0 \\ 1 & t > 1 \end{Bmatrix}$ are indistinguishable. Thus $\mathcal{L}[1] = \mathcal{L}[u(t)] = \int_{0-}^{\infty} e^{-st} \cdot 1\, dt = \left[\frac{e^{-st}}{-s}\right]_{t=0}^{t=\infty} = 0 + \frac{1}{s} = \frac{1}{s}$

   Here we used the fact that for $s > 0$, $\lim_{t\to\infty}\left[e^{-st}\right] = 0$. Indeed, this is still the case even if we permit s to be complex with positive real part, i.e. $\text{Re}(s) > 0$.

2) If $f(t) = t$, we calculate $F(s) = \mathcal{L}[t] = \int_{0-}^{\infty} te^{-st}dt = \left[\frac{te^{-st}}{-s}\right]_{t=0-}^{t=\infty} + \frac{1}{s}\int_{0-}^{\infty} e^{-st}dt = 0 + \frac{1}{s}\mathcal{L}[1] = \frac{1}{s^2}$

3) If $f(t) = t^2$, we calculate $F(s) = \mathcal{L}[t^2] = \int_{0-}^{\infty} t^2 e^{-st}dt = \left[\frac{t^2 e^{-st}}{-s}\right]_{t=0-}^{t=\infty} + \frac{2}{s}\int_{0-}^{\infty} te^{-st}dt = 0 + \frac{2}{s}\mathcal{L}[t] = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3}$

4) ***s*-derivative rule**: $\boxed{\mathcal{L}[tf(t)] = -F'(s)}$. We can establish this by noting that if $F(s) = \mathcal{L}[f(t)] = \int_{0-}^{\infty} e^{-st} f(t)dt$,

   then $F'(s) = \frac{d}{ds}\int_{0-}^{\infty} e^{-st} f(t)dt = -\int_{0-}^{\infty} e^{-st} t\, f(t)dt = -\mathcal{L}[t\, f(t)]$, so $\mathcal{L}[tf(t)] = -F'(s)$.

   From this we see that $\mathcal{L}[t^2] = \mathcal{L}[t \cdot t] = -\frac{d}{ds}\mathcal{L}[t] = -\frac{d}{ds}\left[\frac{1}{s^2}\right] = \frac{2}{s^3}$; $\mathcal{L}[t^3] = \mathcal{L}[t \cdot t^2] = -\frac{d}{ds}\mathcal{L}[t^2] = -\frac{d}{ds}\left[\frac{2}{s^3}\right] = \frac{3!}{s^4}$; $\mathcal{L}[t^4] = \mathcal{L}[t \cdot t^3] = -\frac{d}{ds}\mathcal{L}[t^3] = -\frac{d}{ds}\left[\frac{3!}{s^4}\right] = \frac{4!}{s^5}$; and so on. Generally, $\boxed{\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}}$.

   This, together with linearity, enables us to calculate the Laplace transform of any polynomial function.

5) If $f(t)=e^{at}$ is an exponential function (really $f(t)=u(t)e^{at}$ since we are only concerned with $t\ge 0$),

$$\mathcal{L}[e^{at}]=\int_{0-}^{\infty}e^{-st}e^{at}dt=\int_{0-}^{\infty}e^{-(s-a)t}dt=\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_{t=0-}^{t=\infty}=\frac{1}{s-a}\text{, so }\boxed{\mathcal{L}[e^{at}]=\frac{1}{s-a}}.$$

6) ***s*-shift rule**: $\boxed{\mathcal{L}[e^{rt}f(t)]=F(s-r)}$. To establish this, we calculate

$\mathcal{L}[e^{rt}f(t)]=\int_{0-}^{\infty}e^{-st}e^{rt}f(t)dt=\int_{0-}^{\infty}e^{-(s-r)t}f(t)dt=F(s-r)$ simply by noting the substitution.

7) **Transforming derivatives**: For any generalized function, $\boxed{\mathcal{L}[f'(t)]=sF(s)-f(0-)}$ where $f(0-)$ represents the initial value of $f(t)$. The unusual notation is there because we will be dealing with discontinuous and generalized functions where we may need to distinguish left-hand from right-hand limits. We can establish this ***t*-derivative rule** by noting that $\mathcal{L}[f'(t)]=\int_{0-}^{\infty}e^{-st}f'(t)dt$. If we use Integration by Parts with $u=e^{-st}$ and $dv=f'(t)dt$, we get $du=-se^{-st}dt$ and $v=f(t)$, so

$$\mathcal{L}[f'(t)]=\int_{0-}^{\infty}e^{-st}f'(t)dt=\left[e^{-st}f(t)\right]_{t=0-}^{t=\infty}+s\int_{0-}^{\infty}e^{-st}f(t)dt=[0]+sF(s)=sF(s)-f(0-)$$

For second derivatives, note that $f''(t)=\frac{d}{dt}f'(t)$, so we can apply the above result to get that

$\mathcal{L}[f''(t)]=s\mathcal{L}[f'(t)]-f'(0-)=s(sF(s)-f(0-))-f'(0-)=s^2F(s)-s\cdot f(0-)-f'(0-)$, so

$\boxed{\mathcal{L}[f''(t)]=s^2F(s)-s\,f(0-)-f'(0-)}$.

Continuing, we get that $\boxed{\mathcal{L}[f'''(t)]=s^3F(s)-s^2\,f(0-)-s\,f'(0-)-f''(0-)}$, and so on.

Generally, $\boxed{\mathcal{L}[f^{(n)}(t)]=s^nF(s)-s^{n-1}f(0-)-s^{n-2}f'(0-)-\cdots-f^{(n-1)}(0-)}$.

8) **Transforming the delta function**: One of our most fundamental transforms is $\boxed{\mathcal{L}[\delta(t)]=1}$. This is relatively easy to see once you're comfortable with the integral formalisms concerning the delta function and how they relate to evaluation. Specifically, $\mathcal{L}[\delta(t)]=\int_{0-}^{\infty}e^{-st}\delta(t)dt=e^0=1$ since this is really just evaluation of the function e^{-st} at $t=0$.

9) **Transforming sines and cosines**: $\boxed{\mathcal{L}[\cos(\omega t)]=\frac{s}{s^2+\omega^2}}$ and $\boxed{\mathcal{L}[\sin(\omega t)]=\frac{\omega}{s^2+\omega^2}}$

We can derive each of these independently, but if we use Euler's Formula and linearity we have that:
$\mathcal{L}[e^{i\omega t}]=\mathcal{L}[\cos(\omega t)+i\sin(\omega t)]=\mathcal{L}[\cos(\omega t)]+i\mathcal{L}[\sin(\omega t)]$, and

$$\mathcal{L}[e^{i\omega t}]=\frac{1}{s-i\omega}\left[\frac{s+i\omega}{s+i\omega}\right]=\frac{s+i\omega}{s^2+\omega^2}=\left(\frac{s}{s^2+\omega^2}\right)+i\left(\frac{\omega}{s^2+\omega^2}\right)$$

Taking real and imaginary parts separately we get that $\mathcal{L}[\cos(\omega t)]=\frac{s}{s^2+\omega^2}$ and $\mathcal{L}[\sin(\omega t)]=\frac{\omega}{s^2+\omega^2}$.

We'll add to this list as we go and as the need arises.

Example: Solve the Initial Value Problem $\ddot{x}+3\dot{x}+2x=2e^{-t},\quad x(0)=0,\quad \dot{x}(0)=0$.

Old Faithful Solution: The homogeneous equation $\ddot{x}+3\dot{x}+2x=0$ is easy to solve. Its characteristic polynomial is $p(s)=s^2+3s+2=(s+2)(s+1)$ which yields the two roots $s=-2$ and $s=-1$. This gives the two independent solutions e^{-2t} and e^{-t}, and all homogeneous solutions are of the form $x_h(t)=c_1e^{-2t}+c_2e^{-t}$. Note that both of these homogeneous solutions are <u>transient</u> in the sense that they decay exponentially as *t* increases.

Next, we need to find a particular solution $x_p(t)$ that satisfies the inhomogeneous differential equation. One look at the right-hand-side and we see that the Exponential Response Formula (ERF) won't work – there is resonance. We can, however, use the Resonant Response Formula to get the particular solution $x_p(t)=\dfrac{2te^{-t}}{p'(-1)}=\dfrac{2te^{-t}}{1}=2te^{-t}$, so the general solution is $x(t)=x_h(t)+x_p(t)=c_1e^{-2t}+c_2e^{-t}+2te^{-t}$. Its derivative is $\dot{x}(t)=-2c_1e^{-2t}-c_2e^{-t}-2te^{-t}+2e^{-t}$. Substituting the (rest) initial conditions gives $\left\{\begin{matrix} x(0)=c_1+c_2=0 \\ \dot{x}(0)=-2c_1-c_2+2=0 \end{matrix}\right\}$, and these can be solved to give $c_1=2, c_2=-2$, so the solution is $\boxed{x(t)=2e^{-2t}-2e^{-t}+2te^{-t}}$.

Solving directly by Laplace transform: We calculated the following Laplace transforms:

(1) $\mathcal{L}(e^{kt})=\dfrac{1}{s-k}$ with region of convergence $\text{Re}(s)>k$, so $\mathcal{L}(e^{-2t})=\dfrac{1}{s+2}$.

(2) If the Laplace transform of $x(t)$ is $X(s)$, then the Laplace transforms of its derivatives are $\mathcal{L}(\dot{x}(t))=sX(s)-x(0-)$ and $\mathcal{L}(\ddot{x}(t))=s^2X(s)-s\,x(0-)-\dot{x}(0-)$. In the case of rest initial conditions $x(0-)=\dot{x}(0-)=0$, these are greatly simplified and, in fact $\mathcal{L}(p(D)x)=p(s)X(s)$. Specifically, $\mathcal{L}(\ddot{x}+3\dot{x}+2x)=s^2X(s)+3sX(s)+2X(s)=(s^2+3s+2)X(s)=p(s)X(s)$.

If we now transform the entire differential equation, we get $(s^2+3s+2)X(s)=\dfrac{2}{s+1}$.

We then solve for $X(s)=\dfrac{2}{(s+1)(s^2+3s+2)}=\dfrac{2}{(s+2)(s+1)^2}=\dfrac{A}{s+2}+\dfrac{B}{s+1}+\dfrac{C}{(s+1)^2}$.

There are many good ways to find the unknowns *A*, *B*, and *C*. For example, if we multiply through by the common denominator to clear fractions, we get $2=A(s+1)^2+B(s+1)(s+2)+C(s+2)$. Plugging in the specific values $s=-2$ and $s=-1$ quickly yields that $A=2$ and $C=2$. Plugging in, for example, $s=0$ and using the values for *A* and *C* then yields $B=-2$. So $X(s)=\dfrac{2}{s+2}-\dfrac{2}{s+1}+\dfrac{2}{(s+1)^2}=2\left(\dfrac{1}{s+2}\right)-2\left(\dfrac{1}{s+1}\right)+2\left(\dfrac{1}{(s+1)^2}\right)$.

Consulting our table of common Laplace transforms, we see that $\dfrac{1}{s+2}=\mathcal{L}(e^{-2t})$, $\dfrac{1}{s+1}=\mathcal{L}(e^{-t})$, and $\dfrac{1}{(s+1)^2}=\mathcal{L}(te^{-t})$, so transforming back (using linearity) gives $\boxed{x(t)=2e^{-2t}-2e^{-t}+2te^{-t}}$.

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} e^{-st} f(t)dt$ for $\text{Re}(s) \gg 0$.

1. Linearity: $\mathcal{L}[af(t)+bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] = aF(s)+bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{rt} f(t)] = F(s-r)$.

4. t-shift rule: $\mathcal{L}[f(t-a)] = e^{-as}F(s)$ if $a \geq 0$ and $f(t) = 0$ for $t < 0$.

   This may also be expressed as $\mathcal{L}[f_a(t)] = e^{-as}F(s)$ where $f_a(t) = u(t-a)f(t-a) = \begin{Bmatrix} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{Bmatrix}$.

5. s-derivative rule: $\mathcal{L}[t f(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s) - f(0-)$

$$\mathcal{L}[f''(t)] = s^2F(s) - sf(0-) - f'(0-)$$

$$\mathcal{L}[f^{(n)}(t)] = s^nF(s) - s^{n-1}f(0-) - s^{n-2}f'(0-) - \cdots - f^{(n-1)}(0-)$$

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s)$, $w(t)$ the unit impulse response.

   If $q(t)$ is regarded as the input signal in $p(D)x = q(t)$, $W(s) = \frac{1}{p(s)}$.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s}$$

$$\mathcal{L}[\delta(t)] = 1$$

$$\mathcal{L}[\delta(t-a)] = \mathcal{L}[\delta_a(t)] = e^{-as}$$

$$\mathcal{L}[u(t-a)] = \mathcal{L}[u_a(t)] = \frac{e^{-as}}{s}$$

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

$$\mathcal{L}[t] = \frac{1}{s^2}$$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$$

$$\mathcal{L}[u(t-a)f(t-a)] = e^{-as}F(s)$$

$$\mathcal{L}[u(t-a)f(t)] = e^{-as}\mathcal{L}[f(t+a)]$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}[t\cos(\omega t)] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

$$\mathcal{L}[t\sin(\omega t)] = \frac{2\omega s}{(s^2+\omega^2)^2}$$

$$\mathcal{L}[e^{zt}\cos(\omega t)] = \frac{s-z}{(s-z)^2+\omega^2}$$

$$\mathcal{L}[e^{zt}\sin(\omega t)] = \frac{\omega}{(s-z)^2+\omega^2}$$

Notes by Robert Winters