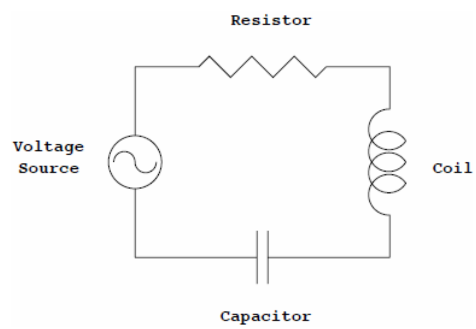
Concourse 18.03 – Lecture #10

We continue the study of mass-spring-dashpot systems and categorize these according to whether they are underdamped, overdamped, or critically damped. We'll solve each of the homogeneous cases by producing linearly independent solutions that span all solutions. Linear differential operators will be introduced to explain why the solutions we derive actually span all solutions. We introduce a few more definitions: zero state, zero state response, and zero input response. We also introduce the Exponential Response Formula (ERF) that we'll soon be using extensively.

Analogy between Mass-Spring-Dashpot systems and LRC circuits

A spring with an attached mass, friction supplied by a dashpot, and external force $F(t)$ is described by the differential equation $\boxed{m\ddot{x}+c\dot{x}+kx=F(t)}$. This purely mechanical system has an electrical analogue known as an LRC circuit where *L* represents the **inductance** associated with a coil, *R* represents the **resistance**, and *C* represents the **capacitance**. Given a voltage source with variable voltage $V(t)$ (measured in volts), the circuit will have at any time a current $I(t)$ (measured in amperes), and the capacitor will be carrying a charge $Q(t)$ (measured in Coulombs).



Series RLC Circuit

In physics, we learn that there are voltage drops associated with each of the elements of the circuit. Specifically, $V_L = L\dot{I} = L\frac{dI}{dt}$ due to the inductance, $V_R = IR$ due to the resistance, and $V_C = Q/C$ due to the capacitance. The sum of the voltage drops must match the voltage source, i.e. $V = V_L + V_R + V_C$. We also know that the current satisfies $\dot{Q} = \frac{dQ}{dt} = I$, so $\ddot{Q} = \frac{dI}{dt} = \dot{I}$ and $\dot{V}_C = \dot{Q}/C = I/C$. If we differentiate to get $\dot{V} = \dot{V}_L + \dot{V}_R + \dot{V}_C$ and substitute the above relations, we get that $\boxed{L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{V}}$ for the rate of change of the applied voltage.

In this mechanical/electrical analogy, the inductance becomes analogous to mass, the resistance is analogous to friction, and the (reciprocal of) capacitance is analogous to the stiffness of the spring. Also the rate of change of voltage is analogous to the external force (which is the rate of change of momentum).

Exponential Response Formula (ERF)

In the previous lecture we introduced a simple method for producing particular solutions to linear constant coefficient ODEs in the case where the input was in the form of an exponential function. This was the Exponential Response Formula. If the ODE is in the form $[p(D)]x(t) = ae^{rt}$ with characteristic polynomial $p(s)$, and if *r* is not a characteristic root, then a particular solution will be $\boxed{x_p(t) = \frac{ae^{rt}}{p(r)}}$. As we noted, this formula will fail in the case where *r* is a characteristic root (since the denominator will vanish). This formula is especially useful for dealing with sinusoidal inputs – either pure sinusoidal inputs or with exponential growth or decay. The key step is to use complex replacement in order to express the input in exponential form.

Example: Find the general solution of the ODE $\ddot{x}+3\dot{x}+2x = 2e^t\cos 3t$.

Solution: The characteristic polynomial is $p(s) = s^2+3s+2 = (s+2)(s+1)$. This gives roots $s_1=-2$, $s_2=-1$, and the homogeneous solutions are of the form $x_h(t) = c_1e^{-2t}+c_2e^{-t}$. To produce a particular solution, we use complex replacement (and then recover the real part). Letting $z(t) = x(t)+i\,y(t)$, we'll simultaneously solve the ODEs $\ddot{x}+3\dot{x}+2x = 2e^t\cos 3t$ and $\ddot{y}+3\dot{y}+2y = 2e^t\sin 3t$. Using Euler's formula, we'll solve the ODE $\ddot{z}+3\dot{z}+2z = 2e^t(\cos 3t + i\sin 3t) = 2e^te^{3it} = 2e^{(1+3i)t}$. Using the Exponential Response Formula, we calculate

$p(1+3i)=(1+3i)^2+3(1+3i)+2=1+6i-9+3+9i+2=-3+15i$, so a particular solution is $z_p(t)=\dfrac{2e^{(1+3i)t}}{-3+15i}$.

We could do one of two things at this point. First, we could multiply the numerator and denominator by the complex conjugate $-3-15i$ and also use Euler's formula to express everything in terms of sines and cosines. This would give:

$$z_p(t)=\frac{2e^{(1+3i)t}}{-3+15i}=\tfrac{1}{117}e^t(-3-15i)(\cos 3t+i\sin 3t)=\tfrac{1}{117}e^t\left[(-3\cos 3t+15\sin 3t)+i(-15\cos 3t-3\sin 3t)\right].$$

We would then recover the real part as $x_p(t)=\tfrac{1}{117}e^t(-3\cos 3t+15\sin 3t)$.

The second option is particularly well suited to the Exponential Response Formula. If we express the denominator as a complex number, i.e. $-3+15i=\sqrt{234}\,e^{i\phi}$ where $\phi=\tan^{-1}\left(\tfrac{15}{-3}\right)=\tan^{-1}(-5)\cong 1.768$ radians (in the 3rd quadrant), we can then write $z_p(t)=\dfrac{2e^{(1+3i)t}}{\sqrt{234}\,e^{i\phi}}=\tfrac{2}{\sqrt{234}}e^t e^{i(3t-\phi)}=\tfrac{2}{\sqrt{234}}e^t\left[\cos(3t-\phi)+i\sin(3t-\phi)\right]$ and recover the real part to give $x_p(t)=\tfrac{2}{\sqrt{234}}e^t\cos(3t-\phi)$. We can then easily see that the <u>gain</u> is $\tfrac{1}{\sqrt{234}}$, the <u>lag</u> is $\phi=\tan^{-1}(-5)\cong 1.768$ and, by writing $x_p(t)=\tfrac{2}{\sqrt{234}}e^t\cos 3(t-\tfrac{1}{3}\phi)$, the <u>time lag</u> is $\tfrac{1}{3}\phi\cong 0.589$.

The general solution may then be expressed as $x(t)=c_1e^{-2t}+c_2e^{-t}+\tfrac{2}{\sqrt{234}}e^t\cos 3(t-\tfrac{1}{3}\phi)$.

Resonance

The case where the Exponential Response Formula fails is when r in the exponential input ae^{rt} is a root of the characteristic polynomial. Though the term "resonance" is perhaps most appropriate when considering sinusoidal inputs with frequency matching the natural frequency of a harmonic oscillator (like a spring), we use the term more generally. Let's understand this situation by considering an example.

Example: Find a particular solution of the ODE $\ddot{x}+3\dot{x}+2x=5e^{-2t}$.

Solution: We cannot use the Exponential Response Formula here because $r=-2$ is a root of the characteristic polynomial $p(s)=s^2+3s+2$. So what do we do? If we think in terms of differential operators, we can express this ODE in the form $(D+2)\circ(D+1)\left[x(t)\right]=5e^{-2t}$ and we know that $(D+2)\left[e^{-2t}\right]=-2e^{-2t}+2e^{-2t}=0$. So, if we apply this differential operator to both sides of the former equation we get:

$$(D+2)\circ(D+2)\circ(D+1)\left[x(t)\right]=(D+2)\left[5e^{-2t}\right]=5(D+2)\left[e^{-2t}\right]=0.$$

So we should seek solutions of the 3rd order homogeneous ODE $(D+2)^2\circ(D+1)\left[x(t)\right]=0$. The characteristic polynomial in this case is $(s+2)^2(s+1)$ which gives the same characteristic roots as before only now the root $s=-2$ has multiplicity 2. This means that the homogeneous solutions are given by $\operatorname{Span}\left\{e^{-2t},te^{-2t},e^{-t}\right\}$. The original inhomogeneous equation already had homogeneous solutions $\operatorname{Span}\left\{e^{-2t},e^{-t}\right\}$, so we seek a particular solution of the form $x_p(t)=Ate^{-2t}$ and use undetermined coefficients. This gives $\dot{x}_p(t)=A(-2t+1)e^{-2t}$ and $\ddot{x}_p(t)=A(4t-4)e^{-2t}$, so $\ddot{x}+3\dot{x}+2x=A(4t-4)e^{-2t}+3A(-2t+1)e^{-2t}+2Ate^{-2t}=-Ae^{-2t}=5e^{-2t}$. Therefore $A=-5$ and the particular solution is $\boxed{x_p(t)=-5te^{-2t}}$.

It's possible to do this in general. Suppose we have an nth order linear ODE in the form $\left[p(D)\right]x(t)=ae^{rt}$ where r is a root with multiplicity k of the characteristic polynomial $p(s)$. This means that we can express the characteristic polynomial as $p(s)=q(s)(s-r)^k$ where $q(s)$ is a polynomial of degree $n-k$. The corresponding differential operator can then be expressed as $T=p(D)=q(D)\circ(D-rI)^k$. If we seek a particular solution of the form $x_p(t)=At^ke^{rt}$, we can calculate

$(D-rI)(At^ke^{rt}) = D(At^ke^{rt}) - rAt^ke^{rt} = A(rt^ke^{rt} + kt^{k-1}e^{rt} - rt^ke^{rt}) = Akt^{k-1}e^{rt}$. If $k \ge 2$, we can apply this operator again to get $(D-rI)^2(At^ke^{rt}) = Ak(k-1)t^{k-2}e^{rt}$. Continuing, we eventually get to $(D-rI)^k(At^ke^{rt}) = Ak(k-1)\cdots(2)(1)e^{rt} = Ak!e^{rt}$. Substituting this into the ODE we get:
$\left[p(D)\right](At^ke^{rt}) = q(D)\circ(D-rI)^k(At^ke^{rt}) = Ak![q(D)(e^{rt})] = Ak!q(r)e^{rt} = ae^{rt}$

So $Ak!q(r) = a$, and $A = \dfrac{a}{k!q(r)}$, and therefore $\boxed{x_p(t) = \dfrac{at^ke^{rt}}{k!q(r)}}$. Though we could just use this as our "Resonant Response Formula", we can differentiate $p(s) = q(s)(s-r)^k$ repeatedly to get $p'(s) = q(s)k(s-r)^{k-1} + q'(s)(s-r)^k$, $p''(s) = q(s)k(k-1)(s-r)^{k-2} + 2q'(s)(s-r)^{k-1} + q''(s)(s-r)^k$ and eventually $p^{(k)}(s) = q(s)k! + (s-r)(\text{polynomial in } s)$, so $p^{(k)}(r) = q(r)k!$. We can therefore in general express the **Resonant Response Formula (RRF)** as $\boxed{x_p(t) = \dfrac{at^ke^{rt}}{p^{(k)}(r)}}$ where $p^{(k)}(r)$ is the value of the kth derivative of the characteristic polynomial evaluated at r. Rarely will we need to use this for $k > 1$, so the usual form is simply $\boxed{x_p(t) = \dfrac{ate^{rt}}{p'(r)}}$. The **Exponential Response Formula (ERF)** is just the $k = 0$ case, i.e. $\boxed{x_p(t) = \dfrac{ae^{rt}}{p(r)}}$

If we had applied the RRF to the previous example, we would have $p(s) = s^2 + 3s + 2$ and we would calculate $p'(s) = 2s + 3$, so $p'(-2) = -1$ and the particular solution would be $x_p(t) = \dfrac{5te^{-2t}}{-1} = -5te^{-2t}$.

Superposition of (particular) solutions
In the case where a linear differential equation has an input expressed as the sum of two or more functions, linearity allows us to find solutions for each input individually and then sum these solutions to produce a solution for the original ODE. That is, if we have a linear ODE of the form $T(f) = g_1 + g_2$ and if can individually find functions f_1 and f_2 such that $T(f_1) = g_1$ and $T(f_2) = g_2$, then since $T(f_1 + f_2) = T(f_1) + T(f_2) = g_1 + g_2$, it follows that $f_1 + f_2$ is a solution to $T(f) = g_1 + g_2$. In fact, the same reason shows that if $T(f_1) = g_1$ and $T(f_2) = g_2$, then $c_1f_1 + c_2f_2$ will be a solution of $T(f) = c_1g_1 + c_2g_2$.

Example: Find a particular solution to the ODE $\ddot{x} + 3\dot{x} + 2x = 5e^{-2t} + t^2$
Solution: We have already solved $\ddot{x} + 3\dot{x} + 2x = 5e^{-2t}$ to get a solution $x_1(t) = -5te^{-2t}$. We can solve $\ddot{x} + 3\dot{x} + 2x = t^2$ using undetermined coefficients and a solution of the form $x(t) = at^2 + bt + c$. This gives

$$2a + 3(2at+b) + 2(at^2+bt+c) = 2at^2 + (6a+2b)t + (2a+3b+2c) = t^2, \text{ so } \left\{\begin{aligned} 2a &= 1 \\ 6a+2b &= 0 \\ 2a+3b+2c &= 0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} a &= \tfrac{1}{2} \\ b &= -\tfrac{3}{2} \\ c &= \tfrac{7}{4} \end{aligned}\right\}. \text{ So}$$

$x_2(t) = \frac{1}{2}t^2 - \frac{3}{2}t + \frac{7}{4}$. Therefore the desired solution is $x_1(t) + x_2(t) = \boxed{-5te^{-2t} + \frac{1}{2}t^2 - \frac{3}{2}t + \frac{7}{4}}$.

Next time we'll look at time invariant linear operators and how we can use this property to find solutions, and we'll develop the method of variation of parameters for finding particular solutions in the case of higher order linear differential equations. We'll also look at the interesting case of resonance where the frequency of a sinusoidal input matches the natural frequency of a harmonic oscillator.

Notes by Robert Winters