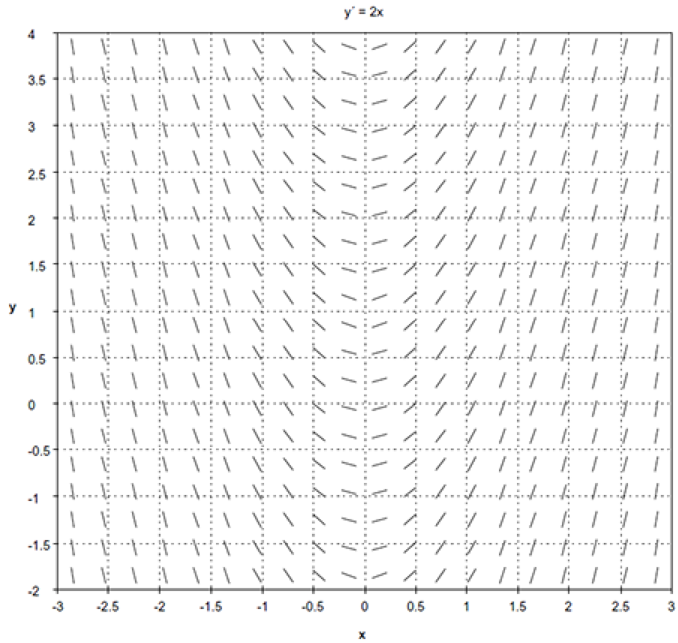
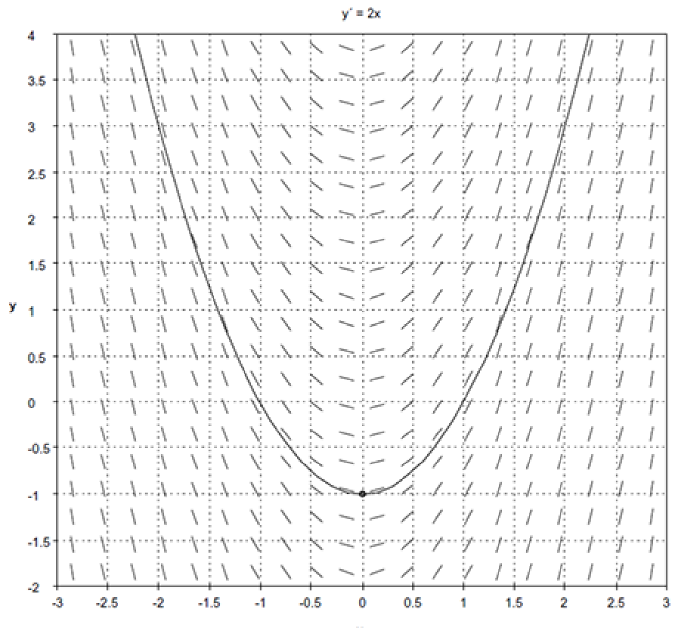
Concourse 18.03 – Lecture #1

Basic notions

There are many contexts in which we seek to discover the relationship between two (or more) variables, e.g. $y = y(x)$ or $x = x(t)$. Such relations are often determined by local information – rates that are measurable or which are given by known formulas. This is the essence of an **ordinary differential equation** (ODE).

For example, for the relation $y = x^2 - 1$, it's the case that $\frac{dy}{dx} = 2x$ and with the additional requirement that $y(0) = -1$, this rate information and the initial condition completely determine the relation $y = x^2 - 1$. The derivative statement is a (first order) **differential equation** and the requirement that the graph pass through the point $(0,-1)$ is called an **initial condition**. We can understand this geometrically by looking at the **slope field** – a drawing in the *xy*-plane that indicates the slope at any point as determined by the given differential equation.





The left image shows only the slope field. The right image indicates the unique solution that matches this slope field and the given initial condition. One important theme that we'll explore will be the conditions under which a given differential equation will yield unique solutions for a given initial condition.

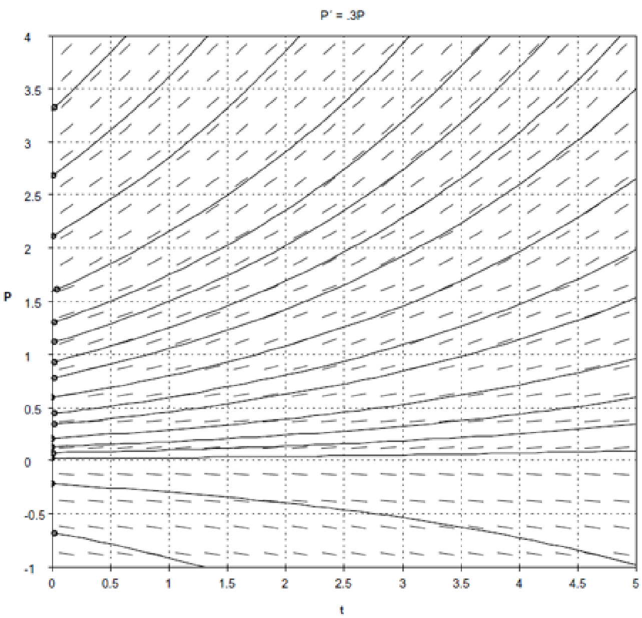
Definition: Given a differential equation in the form $\frac{dy}{dx} = F(x, y)$, an **isocline** is a curve along which the slope is constant.

Isoclines are especially useful for drawing slope fields by hand. For any specific slope, we find the corresponding isocline and pencil in little dashes to indicate that slope along the isocline. If we do this for a range of values for the slope, we can usually get a very good indication of the slope field and of the solutions (also called integral curves) it will yield. Note that in the example above, the isoclines were all vertical lines because $\frac{dy}{dx} = 2x$ will be constant where *x* is constant.

Unrestricted growth

The are many situations from physics to finance in which unrestricted growth at a fixed relative growth rate is the rule. If we express this as a differential equation in how a quantity *P* grows in time *t*, the corresponding differential equation may be expressed in terms of a fixed relative growth rate *k* as $\frac{1}{P}\frac{dP}{dt} = k$ or in terms of absolute growth rate as $\boxed{\frac{dP}{dt} = kP}$. We will presumably also have some initial condition $P(0) = P_0$.

The slope field in this case will have horizontal lines as its isoclines, i.e. $\frac{dP}{dt} = kP = \text{constant} \quad \Rightarrow \quad P = \text{constant}$.



You have most likely already seen an analytic solution to this differential equation. It is an example of a **separable equation** in which we can algebraically separate the variables and integrate both sides of the equation. That is, we can formally write $\frac{dP}{P} = k\,dt$ and integrate to get

$$\int \frac{dP}{P} = \int k\,dt \quad \Rightarrow \quad \ln|P| = kt + c \quad \Rightarrow \quad P = Ae^{kt}$$

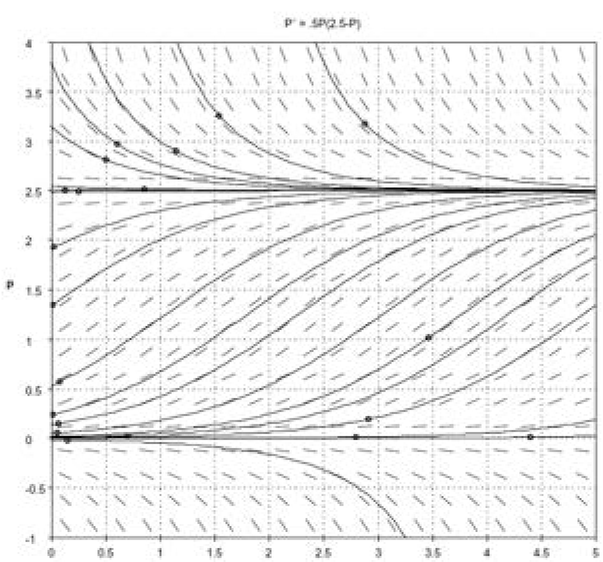
where we've used some basic facts about rules of exponents and absolute value to get the result. Note that this yields exponential growth where the rate *k* appears as the coefficient in the exponent. If we also use the initial condition, we have $P(0) = Ae^0 = A = P_0$, so individual solutions are given by $\boxed{P(t) = P_0 e^{kt}}$. The picture at left indicates the case where $k > 0$ (growth) with several integral curves shown. The $k < 0$ case would give exponential decay.

Differential equations can also involve higher order derivatives. For example, Newton's 2nd Law is usually stated as $F = ma$ where m represents mass, F is the applied force, and a is the acceleration. We know that if x represents position, and v represents velocity, then $v = \frac{dx}{dt}$ and $a = \frac{dv}{dt}$, so Newton's 2nd Law can also be expressed as the **2nd order ordinary differential equation** $\frac{d^2x}{dt^2} = \frac{F}{m} = a$. In the special case of uniform acceleration (or a constant applied force), this is simple to solve. We have $\frac{dv}{dt} = a$ (constant), so $v = at + c_1$. If the initial velocity is $v(0) = v_0$, then $v(0) = c_1 = v_0$, so $\frac{dx}{dt} = v(t) = at + v_0$. One more integration gives $x(t) = \frac{1}{2}at^2 + v_0 t + c_2$, and if the initial position is $x(0) = x_0$ then $x(0) = c_2 = x_0$, so $\boxed{x(t) = \frac{1}{2}at^2 + v_0 t + x_0}$.

Logistic model for growth in a limited environment

In an environment where a population grows with limited resources, it's not realistic to expect unlimited growth. We can model this situation by assuming that the relative growth rate k declines linearly with growing population, at some point (called the carrying capacity) vanishes, and becomes negative when population exceeds this carrying capacity. This is most simply stated as $\frac{1}{P}\frac{dP}{dt} = k(1 - \frac{P}{L})$ or $\frac{dP}{dt} = cP(L - P)$ where $c = \frac{k}{L}$. This is known as the logistic growth model.

The slope field and some trajectories are shown (right) for the differential equation $\frac{dP}{dt} = .5P(2.5 - P)$.



There are two special isoclines for the logistic model, namely the places where $\frac{dP}{dt} = 0$. These occur where $P = 0$ and where $P = L$ and correspond to **equilibria**. If the initial condition lies on either of these lines, the solutions will be constant for all *t*, i.e. $P(t) = 0$ or $P(t) = L$. For initial conditions exceeding *L*, solutions will decay down to the carrying capacity. For any initial condition between 0 and *L*, the solutions will rise and eventually level off at the carrying capacity. Though not meaningful in application, initial values less than 0 will yield solutions that diverge negatively away from 0. We see in this relatively simple model the important distinction between **stable equilibria** (nearby solutions converge toward the equilibrium) and **unstable equilibria** (nearby solutions diverge away from the equilibrium).

Though we can analytically solve the logistic equation (using separation of variables and the method of partial fractions) to give an explicit formula for solutions, the point here is simply that it's often possible to understand qualitatively how the solutions behave just from understanding the slope field – even if we don't produce explicit solutions.

Definition: A differential equation of the form $\frac{d^n x}{dt^n}+p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}}+\cdots+p_1(t)\frac{dx}{dt}+p_0(t)x(t)=q(t)$, where $p_{n-1}(t),\ldots,p_1(t),p_0(t),q(t)$ are functions of the independent variable *t*, is called an ***n*th order linear ordinary differential equation**. In the case where $q(t)=0$ for all *t*, we call the equation **homogeneous**. Otherwise we call it **inhomogeneous**.

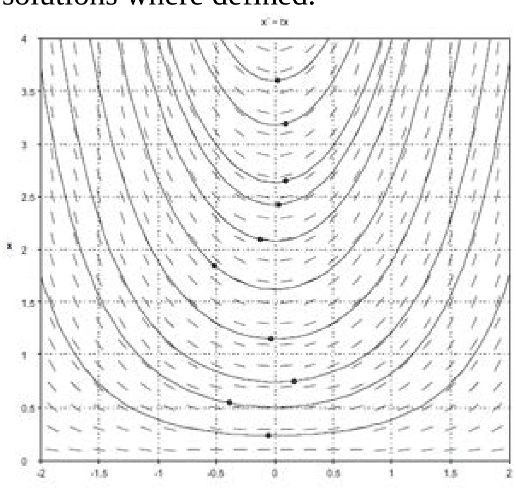
The first three of the previous examples of ODEs (ordinary differential equations) were linear: (a) $\frac{dy}{dx}=2x$ in inhomogeneous; (b) $\frac{dP}{dt}-kP=0$ is homogeneous with constant coefficients; and (c) $\frac{d^2x}{dt^2}=a\ (\text{constant})$ is 2nd order inhomogeneous. The last example given was not linear since it cannot be put into the required form.

In general, any **1st order ODE** can be put in the form $\frac{dy}{dx}=F(x,y)$ for some function $F(x,y)$. If an initial condition $y(a)=b$ is given, we call this an **initial value problem** (IVP). [If we are working with t as the independent variable and x as the dependent variable, we would have $\frac{dx}{dt}=F(t,x)$.]

Question: Under what conditions will this differential equation yield a unique solution for a given initial condition? This is actually <u>two questions</u>: (a) Does a solution exist?; and (b) Is this solution unique?
The answer to these questions is the subject of the following important theorem:

Existence and Uniqueness Theorem: Suppose $F(x,y)$ and the partial derivative F_y are continuous in some rectangle R containing the point $(x_0,y_0)=(a,b)$. Then for some open interval I containing a, the initial value problem $\frac{dy}{dx}=F(x,y)$, $y(a)=b$ has a unique solution defined on the interval I.

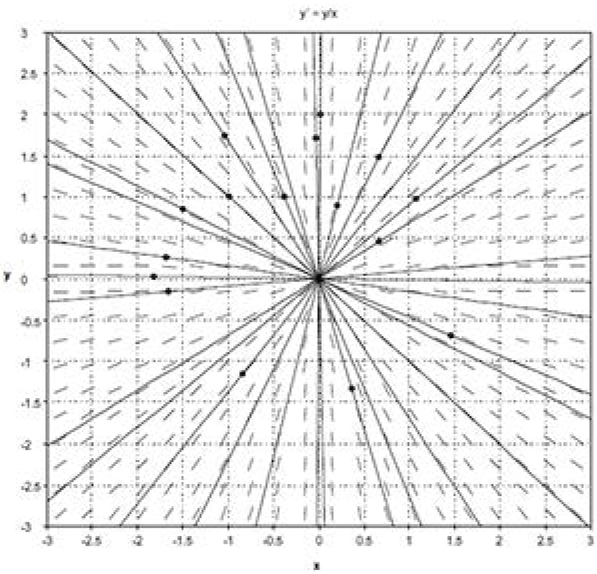
Example: Analyze the ODE $\frac{dx}{dt}=tx$ using its slope field, and solve it analytically to give a formula for all solutions where defined.



Solution: First, note that the right-hand-side is $F(t,x)=tx$ which clearly satisfies the conditions for existence and uniqueness of solutions (for any initial condition).
This is a separable equation. We rewrite $\frac{dx}{dt}=tx$ as $\frac{dx}{x}=tdt$ and integrate $\int\frac{dx}{x}=\int tdt$ to get $\ln|x|=\frac{1}{2}t^2+C$. If we exponentiate both sides and use some familiar algebra rules we get $x(t)=Ae^{\frac{1}{2}t^2}$. If an initial condition is given as $x(0)=x_0$, this will give us the (unique) solution $x(t)=x_0e^{\frac{1}{2}t^2}$.

A proof of the Existence and Uniqueness Theorem can be found in the Appendix of the Edwards & Penney text. The condition that F_y be continuous is actually slightly more restrictive than is necessary to prove the theorem, and proofs in other texts use a milder restriction that requires only that this derivative be bounded in a particular way. Another good source for this theorem is the text by Hirsch, Smale, and Devaney. It is quite technical.

Example: Analyze the ODE $\frac{dy}{dx}=\frac{y}{x}$ using its slope field, and solve it analytically to give a formula for all solutions where defined.

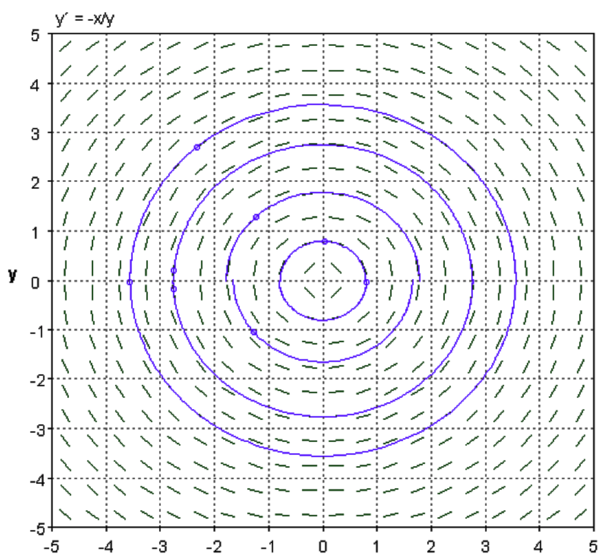


Solution: The slope field for $\frac{dy}{dx}=\frac{y}{x}$ is pretty easy to understand here. Take note, however, that if we have any initial condition where $x=0$, the only integral curve through that point will be a vertical line, so we'll be unable to solve for $y=y(x)$ near such a point. This coincides with the fact that $F(x,y)=\frac{y}{x}$ is discontinuous at any such point. Further note that though there appear to be local solutions passing through any other point, all such solutions pass through (or at least converge toward) the origin.

This ODE is easy to solve: Rewrite $\frac{dy}{dx}=\frac{y}{x}$ as $\frac{dy}{y}=\frac{dx}{x}$ and integrate $\int\frac{dy}{y}=\int\frac{dx}{x}$ to get that $\ln|y|=\ln|x|+C$ and $y=Ax$ for some constant *A*. These are just the lines through the origin that we see in the diagram, and there is no unique solution passing through $(0,0)$.

Orthogonal trajectories: Given any ODE of the form $\frac{dy}{dx}=F(x,y)$, since $\frac{dy}{dx}$ represents the slope at any given point, we can rotate all of these to give orthogonal (perpendicular) slopes that are the negative reciprocals of the original slopes. That is, we would look at the new ODE $\frac{dy}{dx}=-\frac{1}{F(x,y)}$.

Example: Using the previous example, the ODE corresponding to its orthogonal trajectories will be $\frac{dy}{dx}=-\frac{x}{y}$. Analyze this using its slope field and solve it analytically to give a formula for all solutions where defined.



Solution: It should be apparent that the integral curves will be circles everywhere perpendicular to the radial lines of the previous example. The Existence and Uniqueness Theorem will fail where $y=0$, and this corresponds precisely to where these circles would "fold over" between the upper semicircle and lower semicircle. Any solution $y(x)$ will not be extendable beyond such a point.

We can rearrange $\frac{dy}{dx}=-\frac{x}{y}$ as $ydy=-xdx$ and integrate $\int ydy=-\int xdx$ to get $\frac{1}{2}y^2=-\frac{1}{2}x^2+c$ or, more simply, $x^2+y^2=C$. These are the circles mentioned above. They yield solutions $y=\sqrt{C-x^2}$ and $y=-\sqrt{C-x^2}$ and we would use initial conditions to determine the value of *C* and whether we have the upper or lower graph as our solution.

A note on numerical methods

We may go into some detail about this elsewhere or you may learn some of the relevant details in recitation. The videos by Prof. Arthur Mattuck on this subject are highly recommended. The main thing to keep in mind is that the software used to produce slope fields and graphical solutions (integral curves) does not operate via magic or divine guidance. There are specific algorithms like Euler's method, various improved Runge-Kutta methods, or perhaps the Dormand-Prince method that give solutions using various error correction methods to produce relatively accurate graphical solutions.

Notes by Robert Winters