6. Power Series

6A. Power Series Operations

6A-1. Find the radius of convergence for each of the following:

a) \( \sum_{n=0}^{\infty} n x^n \)  
b) \( \sum_{n=0}^{\infty} \frac{x^{2n}}{n^2} \)  
c) \( \sum_{n=1}^{\infty} n! x^n \)  
d) \( \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} x^n \)

6A-2. Starting from the series \( \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \) and \( \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \), by using operations on series (substitution, addition and multiplication, term-by-term differentiation and integration), find series for each of the following:

a) \( \frac{1}{(1-x)^2} \)  
b) \( xe^{-x^2} \)  
c) \( \tan^{-1} x \)  
d) \( \ln(1+x) \)

6A-3. Let \( y = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \). Show that

a) \( y \) is a solution to the ODE \( y'' - y = 0 \)  
b) \( y = \sinh x = \frac{1}{2} (e^x - e^{-x}) \).

6A-4. Find the sum of the following power series (using the operations in 6A-2 as a help):

a) \( \sum_{n=0}^{\infty} x^{3n+2} \)  
b) \( \sum_{n=0}^{\infty} \frac{x^n}{n+1} \)  
c) \( \sum_{n=0}^{\infty} nx^n \)

6B. First-order ODE’s

6B-1. For the nonlinear IVP \( y' = x + y^2, \quad y(0) = 1 \), find the first four nonzero terms of a series solution \( y(x) \) two ways:

a) by setting \( y = \sum_{n=0}^{\infty} a_n x^n \) and finding in order \( a_0, a_1, a_2, \ldots \), using the initial condition and substituting the series into the ODE;

b) by differentiating the ODE repeatedly to obtain \( y(0), y'(0), y''(0), \ldots \), and then using Taylor’s formula.

6B-2. Solve the following linear IVP by assuming a series solution \( y = \sum_{n=0}^{\infty} a_n x^n \), substituting it into the ODE and determining the \( a_n \) recursively by the method of undetermined coefficients. Then sum the series to obtain an answer in closed form, if possible (the techniques of 6A-2,4 will help):

a) \( y' = x + y, \quad y(0) = 0 \)  
b) \( y' = -xy, \quad y(0) = 1 \)  
c) \( (1-x)y' - y = 0, \quad y(0) = 1 \)
6C. Solving Second-order ODE's

6C-1. Express each of the following as a power series of the form $\sum_{N}^{\infty} b_n x^n$. Indicate the value of $N$, and express $b_n$ in terms of $a_n$.

   a) $\sum_{1}^{\infty} a_n x^{n+3}$
   b) $\sum_{0}^{\infty} n(n-1)a_n x^{n-2}$
   c) $\sum_{1}^{\infty} (n+1)a_n x^{n-1}$

6C-2. Find two independent power series solutions $\sum a_n x^n$ to $y'' - 4y = 0$, by obtaining a recursion formula for the $a_n$.

6C-3. For the ODE $y'' + 2xy' + 2y = 0$,

   a) find two independent series solutions $y_1$ and $y_2$;
   b) determine their radius of convergence;
   c) express the solution satisfying $y(0) = 1, y'(0) = -1$ in terms of $y_1$ and $y_2$;
   d) express the series in terms of elementary functions (i.e., sum the series to an elementary function).

(One of the two series is easily recognizable; the other can be gotten using the operations on series, or by using the known solution and the method of reduction of order—see Exercises 2B.)

6C-4. Hermite’s equation is $y'' - 2xy' + ky = 0$. Show that if $k$ is a positive even integer $2m$, then one of the power series solutions is a polynomial of degree $m$.

6C-5. Find two independent series solutions in powers of $x$ to the Airy equation: $y'' = xy$.

   Determine their radius of convergence. For each solution, give the first three non-zero terms and the general term.

6C-6. Find two independent power series solutions $\sum a_n x^n$ to

   $$(1-x^2)y'' - 2xy' + 6y = 0.$$ 

   Determine their radius of convergence $R$. To what extent is $R$ predictable from the original ODE?

6C-7. If the recurrence relation for the $a_n$ has three terms instead of just two, it is more difficult to find a formula for the general term of the corresponding series. Give the recurrence relation and the first three nonzero terms of two independent power series solutions to

   $$y'' + 2y' + (x - 1)y = 0.$$