18.03 EXERCISES

1. First-order ODE's

1A. Introduction; Separation of Variables

1A-1. Verify that each of the following ODE's has the indicated solutions (c_i, a are constants):

a) $y'' - 2y' + y = 0, \qquad y = c_1 e^x + c_2 x e^x$

b) $xy' + y = x \sin x, \qquad y = \dfrac{\sin x + a}{x} - \cos x$

1A-2. On how many arbitrary constants (also called *parameters*) does each of the following families of functions depend? (There can be less than meets the eye. . . ; a, b, c, d, k are constants.)

a) $c_1 e^{kx}$ b) $c_1 e^{x+a}$ c) $c_1 + c_2 \cos 2x + c_3 \cos^2 x$ d) $\ln(ax + b) + \ln(cx + d)$

1A-3. Write down an explicit solution (involving a definite integral) to the following initial-value problems (IVP's):

a) $y' = \dfrac{1}{y^2 \ln x}, \quad y(2) = 0$ b) $y' = \dfrac{ye^x}{x}, \quad y(1) = 1$

1A-4. Solve the IVP's (initial-value problems):

a) $y' = \dfrac{xy + x}{y}, \quad y(2) = 0$ b) $\dfrac{du}{dt} = \sin t \cos^2 u, \quad u(0) = 0$

1A-5. Find the general solution by separation of variables:

a) $(y^2 - 2y)\, dx + x^2 dy = 0$ b) $x\dfrac{dv}{dx} = \sqrt{1 - v^2}$

c) $y' = \left(\dfrac{y - 1}{x + 1}\right)^2$ d) $\dfrac{dx}{dt} = \dfrac{\sqrt{1 + x}}{t^2 + 4}$

1B. Standard First-order Methods

1B-1. Test the following ODE's for exactness, and find the general solution for those which are exact.

a) $3x^2 y\, dx + (x^3 + y^3)\, dy = 0$ b) $(x^2 - y^2)\, dx + (y^2 - x^2)\, dy = 0$

c) $ve^{uv} du + ye^{uv} dv = 0$ d) $2xy\, dx - x^2 dy = 0$

1B-2. Find an integrating factor and solve:

a) $2x\, dx + \dfrac{x^2}{y}\, dy = 0$ b) $y\, dx - (x + y)\, dy = 0, \quad y(1) = 1$

c) $(t^2 + 4)\, dt + t\, dx = x\, dt$ d) $u(du - dv) + v(du + dv) = 0. \quad v(0) = 1$

1B-3. Solve the homogeneous equations

a) $y' = \dfrac{2y-x}{y+4x}$ b) $\dfrac{dw}{du} = \dfrac{2uw}{u^2-w^2}$ c) $xy\,dy - y^2dx = x\sqrt{x^2-y^2}\,dx$

1B-4. Show that a change of variable of the form $u = \dfrac{y}{x^n}$ turns $y' = \dfrac{4+xy^2}{x^2y}$ into an equation whose variables are separable, and solve it.

(Hint: as for homogeneous equations, since you want to get rid of y and y', begin by expressing them in terms of u and x.)

1B-5. Solve each of the following, finding the general solution, or the solution satisfying the given initial condition.

a) $xy' + 2y = x$ b) $\dfrac{dx}{dt} - x\tan t = \dfrac{t}{\cos t}$, $x(0) = 0$

c) $(x^2-1)y' = 1 - 2xy$ d) $3v\,dt = t(dt - dv)$, $v(1) = \frac{1}{4}$

1B-6. Consider the ODE $\dfrac{dx}{dt} + ax = r(t)$, where a is a positive constant, and $\lim\limits_{t\to\infty} r(t) = 0$. Show that if $x(t)$ is any solution, then $\lim\limits_{t\to\infty} x(t) = 0$. (Hint: use L'Hospital's rule.)

1B-7. Solve $y' = \dfrac{y}{y^3+x}$. Hint: consider $\dfrac{dx}{dy}$.

1B-8. The **Bernoulli** equation. This is an ODE of the form $y' + p(x)y = q(x)y^n$, $n \neq 1$. Show it becomes linear if one makes the change of dependent variable $u = y^{1-n}$.

(Hint: begin by dividing both sides of the ODE by y^n .)

1B-9. Solve these Bernoulli equations using the method decribed in 1B-8:

a) $y' + y = 2xy^2$ b) $x^2y' - y^3 = xy$

1B-10. The **Riccati** equation. After the linear equation $y' = A(x) + B(x)\,y$, in a sense the next simplest equation is the Riccati equation

$$y' = A(x) + B(x)y + C(x)y^2,$$

where the right-hand side is now a quadratic function of y instead of a linear function. In general the Riccati equation is not solvable by elementary means. However,

a) show that if $y_1(x)$ is a solution, then the general solution is

$$y = y_1 + u,$$

where u is the general solution of a certain Bernoulli equation (cf. 1B-8).

b) Solve the Riccati equation $y' = 1 - x^2 + y^2$ by the above method.

1B-11. Solve the following second-order autonomous equations ("autonomous" is an important word; it means that the independent variable does not appear explicitly in the equation — it does lurk in the derivatives, of course.)

a) $y'' = a^2y$ b) $yy'' = {y'}^2$ c) $y'' = y'(1+3y^2)$, $y(0) = 1$, $y'(0) = 2$

1B-12. For each of the following, tell what type of ODE it is — i.e., what method you would use to solve it. (Don't actually carry out the solution.) For some, there are several methods which could be used.

1. $(x^3+y)\,dx + x\,dy = 0$
2. $\dfrac{dy}{dt} + 2ty - e^{-t} = 0$
3. $y' = \dfrac{x^2-y^2}{5xy}$
4. $(1+2p)\,dq + (2-q)\,dp = 0$
5. $\cos x\,dy = (y\sin x + e^x)\,dx$
6. $x(\tan y)y' = -1$
7. $y' = \dfrac{y}{x} + \dfrac{1}{y}$
8. $\dfrac{dv}{du} = e^{2u+3v}$
9. $xy' = y + xe^{y/x}$
10. $xy' - y = x^2 \sin x$
11. $y' = (x+e^y)^{-1}$
12. $y' + \dfrac{2y}{x} - \dfrac{y^2}{x} = 0$
13. $\dfrac{dx}{dy} = -x\left(\dfrac{2x^2y + \cos y}{3x^2y^2 + \sin y}\right)$
14. $y' + 3y = e^{-3t}$
15. $x\dfrac{dy}{dx} - y = \sqrt{x^2+y^2}$
16. $\dfrac{y'-1}{x^2} = 1$
17. $xy' - 2y + y^2 = x^4$
18. $y'' = \dfrac{y(y+1)}{y'}$
19. $t\dfrac{ds}{dt} = s(1 - \ln t + \ln s)$
20. $\dfrac{dy}{dx} = \dfrac{3-2y}{2x+y+1}$
21. $x^2y' + xy + y^2 = 0$
22. $y'\tan(x+y) = 1 - \tan(x+y)$
23. $y\,ds - 3s\,dy = y^4\,dy$
24. $du = -\dfrac{1+u\cos^2 t}{t\cos^2 t}\,dt$
25. $y' + y^2 + (2x+1)y + 1 + x + x^2 = 0$
26. $y'' + x^2y' + 3x^3 = \sin x$

1C. Graphical and Numerical Methods

1C-1. For each of the following ODE's, draw a direction field by using about five isoclines; the picture should be square, using the intervals between -2 and 2 on both axes. Then sketch in some integral curves, using the information provided by the direction field. Finally, do whatever else is asked.

a) $y' = -\dfrac{y}{x}$; solve the equation exactly and compare your integral curves with the correct ones.

b) $y' = 2x + y$; find a solution whose graph is also an isocline, and verify this fact analytically (i.e., by calculation, not from a picture).

c) $y' = x - y$; same as in (b).

d) $y' = x^2 + y^2 - 1$

e) $y' = \dfrac{1}{x+y}$; use the interval -3 to 3 on both axes; draw in the integral curves that pass respectively through $(0,0)$, $(-1,1)$, $(0,-2)$. Will these curves cross the line $y = -x - 1$? Explain by using the Intersection Principle (Notes G, (3)).

1C-2. Sketch a direction field, concentrating on the first quadrant, for the ODE

$$y' = \frac{-y}{x^2 + y^2} \ .$$

Explain, using it and the ODE itself how one can tell that the solution $y(x)$ satisfying the initial condition $y(0) = 1$

a) is a decreasing function for $y > 0$;
b) is always positive for $x > 0$.

1C-3. Let $y(x)$ be the solution to the IVP $y' = x - y, \quad y(0) = 1.$

a) Use the Euler method and the step size $h = .1$ to to find an approximate value of $y(x)$ for $x = .1, .2, .3$. (Make a table as in notes G).
Is your answer for $y(.3)$ too high or too low, and why?

b) Use the Modified Euler method (also called Improved Euler, or Heun's method) and the step size $h = .1$ to determine the approximate value of $y(.1)$. Is the value for $y(.1)$ you found in part (a) corrected in the right direction — e.g., if the previous value was too high, is the new one lower?

1C-4. Use the Euler method and the step size $.1$ on the IVP $y' = x + y^2, \quad y(0) = 1,$ to calculate an approximate value for the solution $y(x)$ when $x = .1, .2, .3$. (Make a table as in Notes G.) Is your answer for $y(.3)$ too high or too low?

1C-5. Prove that the Euler method converges to the exact value for $y(1)$ as the progressively smaller step sizes $h = 1/n, \quad n = 1, 2, 3, \ldots$ are used, for the IVP

$$y' = x - y, \qquad y(0) = 1 \ .$$

(First show by mathematical induction that the approximation to y(1) gotten by using the step size $1/n$ is

$$y_n = 2(1 - h)^n - 1 + nh \ .$$

The exact solution is easily found to be $y = 2e^{-x} + x - 1$.)

1C-6. Consider the IVP $y' = f(x), \quad y(0) = y_0.$

We want to calculate $y(2nh)$, where h is the step size, using n steps of the Runge-Kutta method.

The exact value, by Chapter D of the notes, is $y(2nh) = y_0 + \displaystyle\int_0^{2nh} f(x)\, dx$.

Show that the value for $y(2nh)$ produced by Runge-Kutta is the same as the value for $y(2nh)$ obtained by using Simpson's rule to evaluate the definite integral.

1C-7. According to the existence and uniqueness theorem, under what conditions on $a(x), b(x)$, and (x) will the IVP

$$a(x)\, y' + b(x)\, y = c(x), \quad y(x_0) = y_0$$

have a unique solution in some interval $[x_0 - h, x_0 + h]$ centered around x_0?

1D. Geometric and Physical Applications

1D-1. Find all curves $y = y(x)$ whose graphs have the indicated geometric property. (Use the geometric property to find an ODE satisfied by $y(x)$, and then solve it.)

a) For each tangent line to the curve, the segment of the tangent line lying in the first quadrant is bisected by the point of tangency.

b) For each normal to the curve, the segment lying between the curve and the x-axis las constant length 1.

c) For each normal to the curve, the segment lying between the curve and the x-axis is bisected by the y-axis.

d) For a fixed a, the area under the curve between a and x is proportional to $y(x) - y(a)$.

1D-2. For each of the following families of curves,

(i) find the ODE satisfied by the family (i.e., having these curves as its integral curves);
(ii) find the orthogonal trajectories to the given family;
(iii) sketch both the original family and the orthogonal trajectories.

a) all lines whose y-intercept is twice the slope
b) the exponential curves $y = ce^x$
c) the hyperbolas $x^2 - y^2 = c$
d) the family of circles centered on the y-axis and tangent to the x-axis.

1D-3. Mixing A container holds V liters of salt solution. At time $t = 0$, the salt concentration is c_0 g/liter. Salt solution having concentration c_1 is added at the rate of k liters/min, with instantaneous mixing, and the resulting mixture flows out of the container at the same rate. How does the salt concentration in the tank vary with time?

Let $x(t)$ be the *amount* of salt in the tank at time t. Then $c(t) = \dfrac{x(t)}{V}$ is the concentration of salt at time t.

a) Write an ODE satisfied by $x(t)$, and give the initial condition.

b) Solve it, assuming that it is pure water that is being added. (Lump the constants by setting $a = k/V$.)

c) Solve it, assuming that c_1 is constant; determine $c(t)$ and find $\lim_{t\to\infty} c(t)$. Give an intuitive explanation for the value of this limit.

d) Suppose now that c_1 is not constant, but is decreasing exponentially with time:

$$c_1 = c_0 e^{-\alpha t}, \qquad \alpha > 0.$$

Assume that $a \neq \alpha$ (cf. part (b)), and determine $c(t)$, by solving the IVP. Check your answer by putting $\alpha = 0$ and comparing with your answer to (c).

1D-4. Radioactive decay A radioactive substance **A** decays into **B**, which then further decays to **C**.

a) If the decay constants of **A** and **B** are respectively λ_1 and λ_2 (the decay constant is by definition ($\ln 2$/half-life)), and the initial amounts are respectively A_0 and B_0, set up an ODE for determining $B(t)$, the amount of **B** present at time t, and solve it. (Assume $\lambda_1 \neq \lambda_2$.)

b) Assume $\lambda_1 = 1$ and $\lambda_2 = 2$. Tell when $B(t)$ reaches a maximum.

1D-5. Heat transfer According to Newton's Law of Cooling, the rate at which the temperature T of a body changes is proportional to the difference between T and the external temperature.

At time $t = 0$, a pot of boiling water is removed from the stove. After five minutes, the water temperature is 80^oC. If the room temperature is 20^oC, when will the water have cooled to 60^oC? (Set up and solve an ODE for $T(t)$.)

1D-6. Motion A mass m falls through air under gravity. Find its velocity $v(t)$ and its terminal velocity (that is, $\lim_{t\to\infty} v(t)$) assuming that

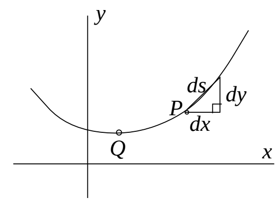
a) air resistance is kv (k constant; this is valid for small v);

b) air resistance is kv^2 (k constant; this is valid for high v).

Call the gravitational constant g. In part (b), lump the constants by introducing a parameter $a = \sqrt{gm/k}$.

1D-7. A loaded cable is hanging from two points of support, with Q the lowest point on the cable. The portion QP is acted on by the total load W on it, the constant tension T_Q at Q, and the variable tension T at P. Both W and T vary with the point P.

Let s denote the length of arc QP.



a) Show that $\dfrac{dx}{T_Q} = \dfrac{dy}{W} = \dfrac{ds}{T}$.

b) Deduce that if the cable hangs under its own weight, and $y(x)$ is the function whose graph is the curve in which the cable hangs, then

(i) $y'' = k\sqrt{1 + y'^2}$, $\quad$ k constant

(ii) $y = \sqrt{s^2 + c^2} + c_1$, $\quad$ c, c_1 constants

c) Solve the suspension bridge problem: the cable is of negligible weight, and the loading is of constant horizontal density. ("Solve" means: find $y(x)$.)

d) Consider the "Marseilles curtain" problem: the cable is of negligible weight, and loaded with equally and closely spaced vertical rods whose bottoms lie on a horizontal line. (Take the x-axis as the line, and show $y(x)$ satisfies the ODE $y'' = k^2y$.)

1E. First-order autonomous ODE's

1E-1. For each of the following autonomous equations $dx/dt = f(x)$, obtain a qualitative picture of the solutions as follows:

(i) draw horizontally the axis of the dependent variable x, indicating the critical points of the equation; put arrows on the axis indicating the direction of motion between the critical points; label each critical point as stable, unstable, or semi-stable. Indicate where this information comes from by including in the same picture the graph of $f(x)$, drawn in dashed lines;

(ii) use the information in the first picture to make a second picture showing the tx-plane, with a set of typical solutions to the ODE: the sketch should show the main qualitative features (e.g., the constant solutions, asymptotic behavior of the non-constant solutions).

a) $x' = x^2 + 2x$
b) $x' = -(x - 1)^2$
c) $x' = 2x - x^2$
d) $x' = (2 - x)^3$

M.I.T. 18.03 Ordinary Differential Equations
18.03 Notes and Exercises