Concourse 18.03 – Fourier III

In this lecture we'll apply Fourier series to solve linear time-invariant ODEs with periodic inputs – specifically the case of harmonic response. We'll also find ways to manipulating known Fourier series to obtain new Fourier series representations of periodic functions.

Harmonic Response to Periodic Inputs

If we couple the Fourier series representation of a periodic input with our linearity methods, we can produce series representations of solutions to linear time-independent (LTI) differential equations.

Example: Find the general solution to the differential equation $\ddot{x}+4x=sq(t)$, where $sq(t)$ is the square-wave function.

Solution: The system corresponds to a harmonic oscillator. The characteristic polynomial is $p(s)=s^2+4$ with characteristic roots $s=\pm 2i$, and the homogeneous solutions are of the form $x_h(t)=c_1\cos 2t+c_2\sin 2t$.

For a particular solution, we use linearity. Using the Fourier series representation $sq(t)\sim\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{2n+1}$, we can individually solve the ODE $\ddot{x}+4x=\sin(2n+1)t$ for each n and then use linearity to reassemble the overall solution (superposition). To do this we use complex replacement and solve $\ddot{z}+4z=e^{i(2n+1)t}$ using the Exponential Response Formula (ERF), and then extract the imaginary part.

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We have $p(i(2n+1)) = 4-(2n+1)^2$, so $\frac{e^{i(2n+1)t}}{4-(2n+1)^2} = \frac{\cos(2n+1)t + i\sin(2n+1)t}{4-(2n+1)^2}$ is a solution, and we extract its imaginary part to get $\frac{\sin(2n+1)t}{4-(2n+1)^2}$.

Using linearity for the ODE $\ddot{x}+4x = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{2n+1}$, we appropriately scale the individual terms and sum to get the particular solution $x_p(t) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{(2n+1)[4-(2n+1)^2]}$.

Note: You will want to be especially careful to make sure any indexed quantities are *inside* the summation. Only actual constants can be factored outside the summation.

If we expand this to show the first few terms, we have

$$x_p(t) = \frac{4}{\pi}\left[\tfrac{1}{3}\sin t - \tfrac{1}{15}\sin 3t - \tfrac{1}{105}\sin 5t - \tfrac{1}{315}\sin 7t - \tfrac{1}{693}\sin 9t - \cdots\right].$$

Note how the amplitudes of the higher frequencies decrease rapidly. As always, the general solution is $x(t) = x_h(t) + x_p(t)$.

More generally, we could solve $\ddot{x}+\omega^2 x = sq(t)$ to get

$$x_p(t) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{(2n+1)[\omega^2-(2n+1)^2]}.$$

This will usually yield a convergent series, but we have a problem in the case where ω is an odd integer since one term of the series will "blow up" in that case. This is a case of resonance and we'll look at that case shortly.

Another Fourier Series calculation

Problem: Find the Fourier series for the function $f(t)=\begin{Bmatrix} 0 & t\in[-\pi,0) \\ t & t\in[0,\pi) \end{Bmatrix}$, extended periodically for all t.

Solution: This function is neither symmetric nor antisymmetric, so we have to compute all the Fourier coefficients.

$$a_0=\tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt=\tfrac{1}{\pi}\int_0^{\pi} t\,dt=\tfrac{1}{\pi}\left[\tfrac{t^2}{2}\right]_0^{\pi}=\tfrac{\pi}{2}$$

$$a_n=\tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt=\tfrac{1}{\pi}\int_0^{\pi} t\cos nt\,dt=\begin{Bmatrix} 0 & n \text{ even} \\ -\frac{2}{\pi n^2} & n \text{ odd} \end{Bmatrix}$$

after a little integration by parts.

$$b_n=\tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt=\tfrac{1}{\pi}\int_0^{\pi} t\sin nt\,dt=\frac{(-1)^{n+1}}{n}$$ after a little integration by parts.

You should carry out these calculations as an exercise.

So $$\boxed{f(t)\sim\tfrac{\pi}{4}-\tfrac{2}{\pi}\sum_{n\text{ odd}}\left(\frac{\cos nt}{n^2}\right)+\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n}\sin nt\right)}$$

Curiosity: Note that for this function $t=0$ is a point of continuity and $f(0)=0$, so

$$\tfrac{\pi}{4}-\tfrac{2}{\pi}\sum_{n\text{ odd}}\left(\frac{1}{n^2}\right)=0\,.$$

Therefore $\displaystyle\sum_{n\text{ odd}}\left(\frac{1}{n^2}\right)=1+\tfrac{1}{9}+\tfrac{1}{25}+\tfrac{1}{49}+\tfrac{1}{81}+\cdots=\tfrac{\pi^2}{8}$ (which we have previously shown).

Tips & Tricks – Manipulation of Fourier series

Different period: We developed our Fourier series representation for functions with a standard period 2π and fundamental interval $[-\pi,\pi]$. If we instead have a function $f(t)$ with period $2L$ and fundamental interval $[-L,L]$, we can simply change variables to produce the corresponding Fourier series in this case. We let $u=\tfrac{\pi t}{L}$ (so $t=\tfrac{Lu}{\pi}$) and define $g(u)=f\left(\tfrac{Lu}{\pi}\right)$ with period 2π and fundamental interval $[-\pi,\pi]$. The Fourier series for $g(u)$ is then:

$$g(u)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nu+b_n\sin nu\right)$$

If we then use the substitution $u=\tfrac{\pi t}{L}$ (and $du=\tfrac{\pi}{L}dt$), we'll have

$$a_0=\tfrac{1}{\pi}\int_{-\pi}^{\pi}g(u)dt=\tfrac{1}{L}\int_{-L}^{L}g(\tfrac{\pi t}{L})\,dt=\tfrac{1}{L}\int_{-L}^{L}f(u)\,du=\tfrac{1}{L}\int_{-L}^{L}f(t)\,dt\,,$$

$$a_n=\tfrac{1}{\pi}\int_{-\pi}^{\pi}g(u)\cos nu\,du=\tfrac{1}{L}\int_{-L}^{L}g(\tfrac{\pi t}{L})\cos(\tfrac{n\pi t}{L})\,dt=\tfrac{1}{L}\int_{-L}^{L}f(t)\cos(\tfrac{n\pi t}{L})\,dt\,,$$

$$b_n=\tfrac{1}{\pi}\int_{-\pi}^{\pi}g(u)\sin nu\,du=\tfrac{1}{L}\int_{-L}^{L}g(\tfrac{\pi t}{L})\sin(\tfrac{n\pi t}{L})\,dt=\tfrac{1}{L}\int_{-L}^{L}f(t)\sin(\tfrac{n\pi t}{L})\,dt\,,$$ and we can write:

$$\boxed{f(t) = g(\tfrac{n\pi t}{L}) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(\tfrac{n\pi t}{L}) + b_n \sin(\tfrac{n\pi t}{L})\right)}$$

Fourier series can be differentiated or integrated term-by-term to produce other Fourier series:

Example: If we start with $sq(t) = \begin{cases} -1 & t \in [-\pi, 0) \\ +1 & t \in [0, \pi) \end{cases} \sim \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n}$ and integrate term-by-term, we get $F(t) \sim \frac{4}{\pi} \sum_{n \text{ odd}} -\frac{\cos nt}{n^2} + C$. If we also insist that $F(0) = 0$ and that $F(t)$ be continuous, we get that $-\frac{4}{\pi}\left(\sum_{n \text{ odd}} \frac{1}{n^2}\right) + C = -\frac{4}{\pi}\left(\frac{\pi^2}{8}\right) + C = -\frac{\pi}{2} + C = 0$, so $C = \frac{\pi}{2}$. This gives $F(t) = |t| = \begin{cases} -t & t \in [-\pi, 0) \\ +t & t \in [0, \pi) \end{cases} \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos nt}{n^2}$, extended periodically for all *t*, a "sawtooth function".

This series could also have been calculated directly using the formulas for the Fourier coefficients and some integration by parts.

Fourier series can be scaled, shifted, etc. to produce other Fourier series

Example #1: Start with $sq(t) = \begin{cases} -1 & t \in [-\pi, 0) \\ +1 & t \in [0, \pi) \end{cases} \sim \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n}$.

Then $1 + sq(t) = \begin{cases} 0 & t \in [-\pi, 0) \\ 2 & t \in [0, \pi) \end{cases} \sim 1 + \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n}$.

So $\frac{1}{2}\left[1+sq(t)\right]=\begin{cases}0 & t\in[-\pi,0)\\ 1 & t\in[0,\pi)\end{cases} \sim \frac{1}{2}+\frac{2}{\pi}\sum_{n\text{ odd}}\frac{\sin nt}{n}$, extended periodically for all t, a different sort of square-wave function.

Example #2: Find the Fourier series for the function $f(t)=\cos(t-\pi/3)$.

Solution: This function is periodic with period 2π. There's no need to consider the formulas for the Fourier coefficients. Simply note that

$f(t)=\cos(t-\pi/3)=\cos t\,\cos(\pi/3)+\sin t\,\sin(\pi/3)=\frac{1}{2}\cos t+\frac{\sqrt{3}}{2}\sin t$.

Harmonic response with resonance

One of the more interesting aspects of using Fourier Series is analyzing how a linear time-independent ODE with a periodic signal yields a response that exhibits **resonance**. The basic idea is that if we expand a periodic signal in a Fourier Series, it's sometimes the case that a single term in the series may be responsible for resonance. The signal may be composed of a whole range of frequencies, but one of them may produce resonance that may be the dominant feature of the response.

Suppose we wish to solve the ODE $\ddot{x}+\omega^2 x = sq(t)\sim\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{2n+1}$, where $sq(t)$ is the square-wave function. We previously observed that this would yield the series solution:

$$x_p(t)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{(2n+1)[\omega^2-(2n+1)^2]}$$

There is a catch, however. All of the terms in the series make sense unless ω is an odd integer. If this is the case, then all but one of the terms in the series will continue to make sense, but we'll have to treat the one term where $\omega = 2n+1$ differently. Let's consider a specific example.

Example: Find a particular solution to the ODE $\ddot{x} + 9x = sq(t)$.

In this case, all of the terms in the above series are as stated, but we have to deal with the $n = 1$ term separately since $\omega = 3$. For this one term we separately solve the ODE $\ddot{x} + 9x = \frac{4}{3\pi}\sin 3t$. If we use complex replacement and later extract the imaginary part, we'll be solving the ODE $\ddot{z} + 9z = \frac{4}{3\pi}e^{3it}$. Since the characteristic polynomial is $p(s) = s^2 + 9$ and $s = 3i$ is a characteristic root, we must use the <u>Resonant Response Formula</u>, i.e. $z = \dfrac{\frac{4}{3\pi}te^{3it}}{p'(3i)}$. Since $p'(s) = 2s$ and $p'(3i) = 6i$, we have the (complex) solution $z = \dfrac{\frac{4}{3\pi}te^{3it}}{6i} = \frac{2}{9\pi}t(-i)[\cos 3t + i\sin 3t] = \frac{2}{9\pi}t[\sin 3t - i\cos 3t]$. Extracting the imaginary part gives $x_3(t) = -\frac{2}{9\pi}t\cos 3t$. This term can then be added into the previous sum to replace the $n = 1$ term. Note, however, that this term is oscillatory but its amplitude grows linearly in time. This is exactly the sort of thing we would expect when the system has resonance – even if it is caused by just one resonant frequency embedded among others.