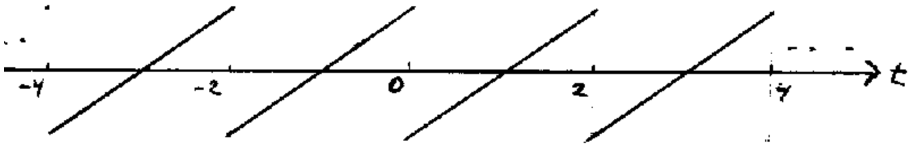
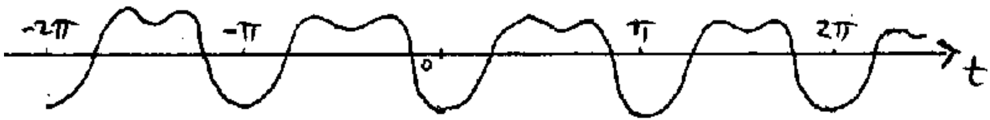
Periodic Inputs and Fourier Series

The solution of an ODE of the form $[p(D)]x(t) = a\cos kt$ or $[p(D)]x(t) = a\sin kt$ is now relatively straightforward through the use of complex replacement, the Exponential Response Formula, and, when needed, the Resonant Response Formula. How might we solve an ODE of the form $[p(D)]x(t) = f(t)$ where $f(t)$ is some other periodic function such as:



or



The way we'll handle this is to successively approximate any such periodic function as a sum of trigonometric functions, solve term-by-term, and then reassemble a solution using linearity (superposition). The approximation method involves **Fourier Series**.

Definition: A function $f(t)$ is called **periodic** with period T if $f(t+nT)=f(t)$ for all t and all integers n. We say that T is the **base period** if it is the least such $T>0$.

Examples: The functions $\sin t$ and $\cos t$ are both periodic with base period 2π. The functions $\sin \omega t$ and $\cos \omega t$ are both periodic with base period $\frac{2\pi}{\omega}$.

Note: Any constant function is also periodic, but with no base period.

For the sake of simplicity, we'll begin by considering periodic functions with base period 2π. We will later rescale to adapt our methods to other base periods. Our methods will be based on the following theorem:

Theorem (Fourier): Suppose a function $f(t)$ is periodic with base period 2π and continuous except for a finite number of jump discontinuities. Then $f(t)$ may be represented by a (convergent) Fourier Series:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

where: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt$, and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\, dt$.

The numbers $\{a_0, a_1, b_1, \cdots, a_n, b_n, \cdots\}$ are called the **Fourier coefficients** of the function $f(t)$.

This representation is an equality at all points of continuity of the function $f(t)$. At any point of discontinuity $t = a$, the series converges to the average of $f(a^-)$ and $f(a^+)$, i.e. the value $\frac{1}{2}[f(a^-) + f(a^+)]$.

Note: (a) If $f(t)$ is an even function [$f(-t) = f(t)$ for all *t*], then $b_n = 0$ for all *n* by basic facts from calculus.

(b) If $f(t)$ is an odd function [$f(-t) = -f(t)$ for all *t*], then $a_0 = 0$ and $a_n = 0$ for all *n* by basic facts from calculus.

Example (Square wave function): $f(t) = sq(t) = \begin{cases} -1 & t \in [-\pi, 0) \\ +1 & t \in [0, \pi) \end{cases}$, extended periodically for all *t*.

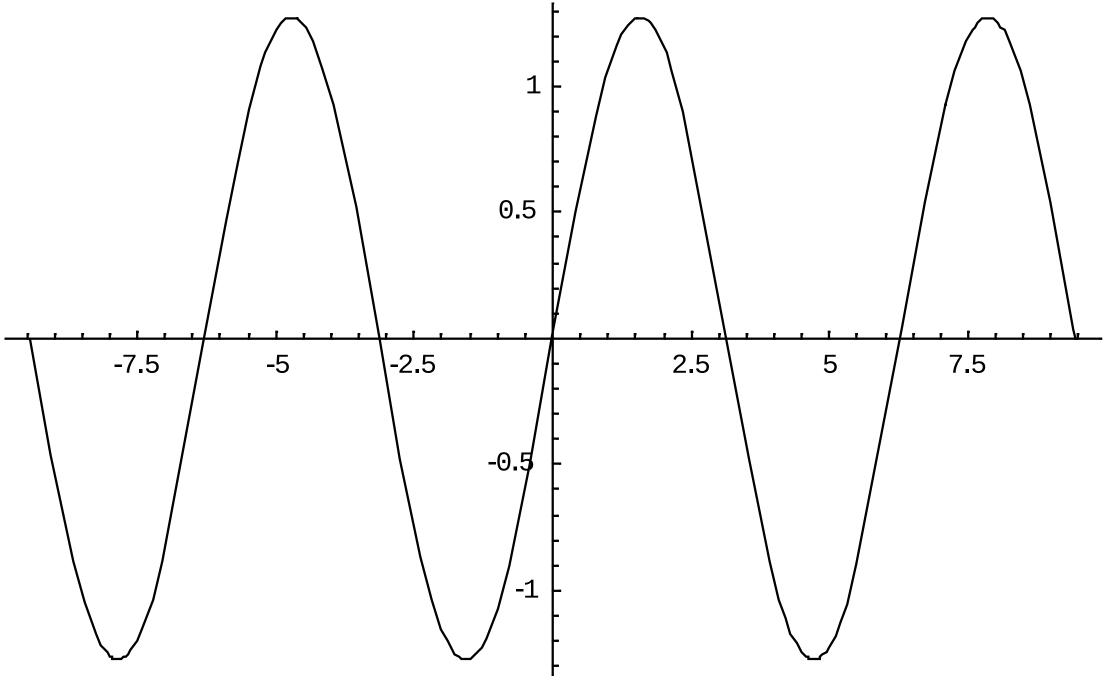
This function is periodic (with period 2π) and antisymmetric, i.e. an odd function. Therefore $a_0 = 0$ and $a_n = 0$ for all n. We calculate

$$b_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \tfrac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin nt\,dt + \int_{0}^{\pi}\sin nt\,dt\right] = \tfrac{1}{\pi}\left[\left[\frac{\cos nt}{n}\right]_{-\pi}^{0} - \left[\frac{\cos nt}{n}\right]_{0}^{\pi}\right]$$

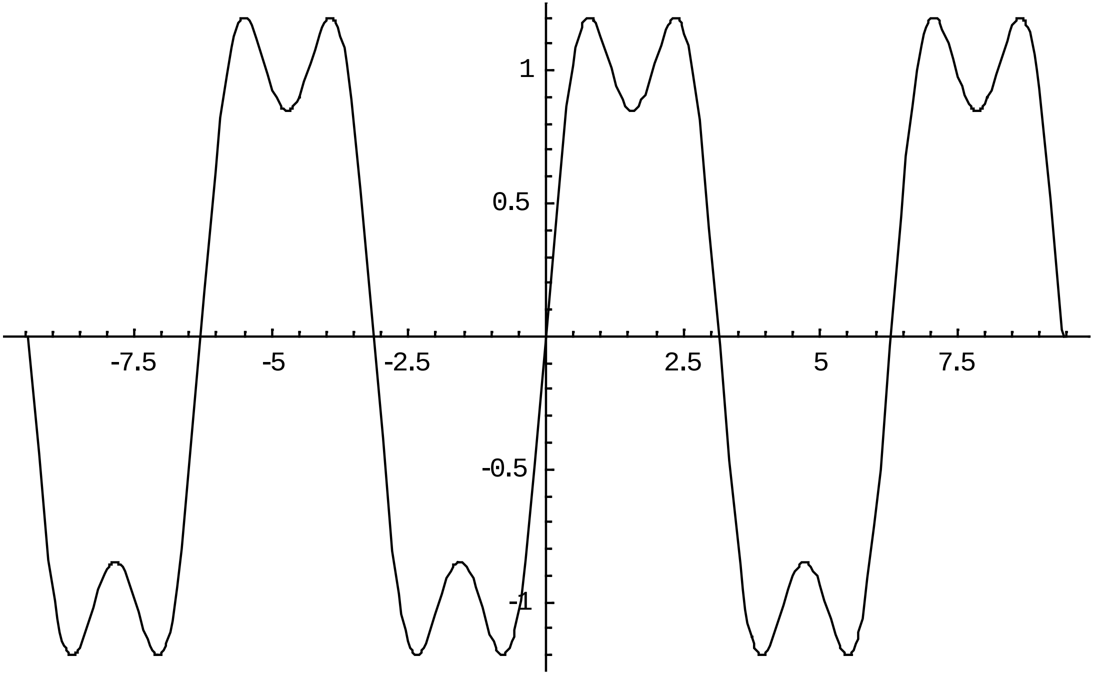
$$= \tfrac{1}{n\pi}\left[\left[1-(-1)^n\right]-\left[(-1)^n-1\right]\right] = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}.$$

So $\boxed{sq(t) \sim \tfrac{4}{\pi}\sum_{n\text{ odd}} \frac{\sin nt}{n} = \tfrac{4}{\pi}[\sin t + \tfrac{1}{3}\sin 3t + \tfrac{1}{5}\sin 5t + \cdots]}$.

The nature of the convergence of this Fourier series toward the square wave function can be seen by graphing the partial sums:



n=1



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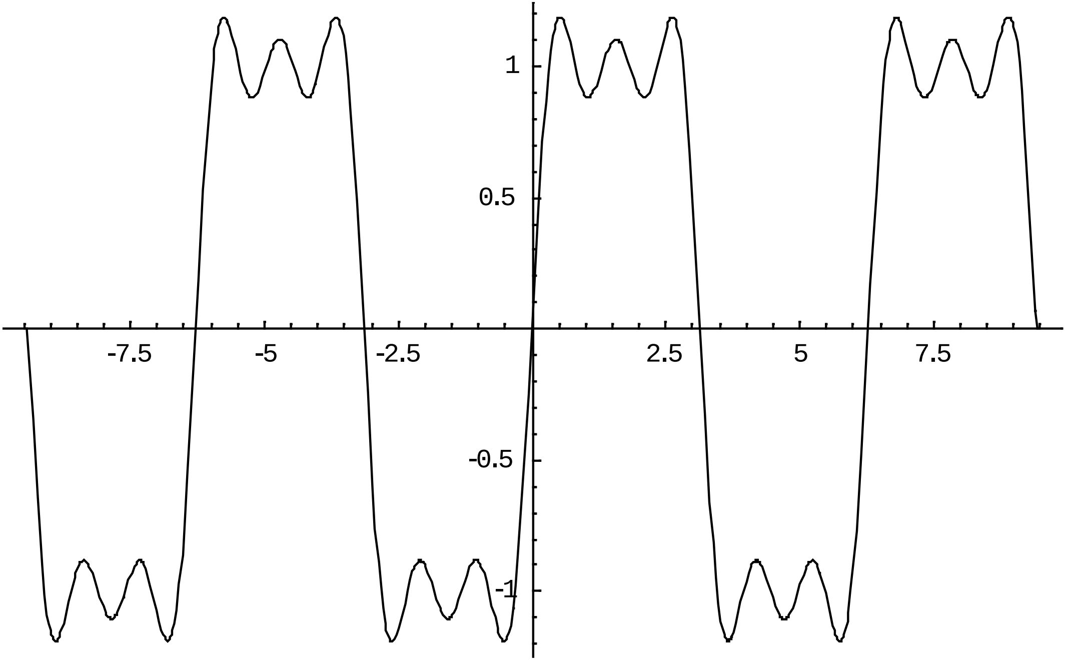

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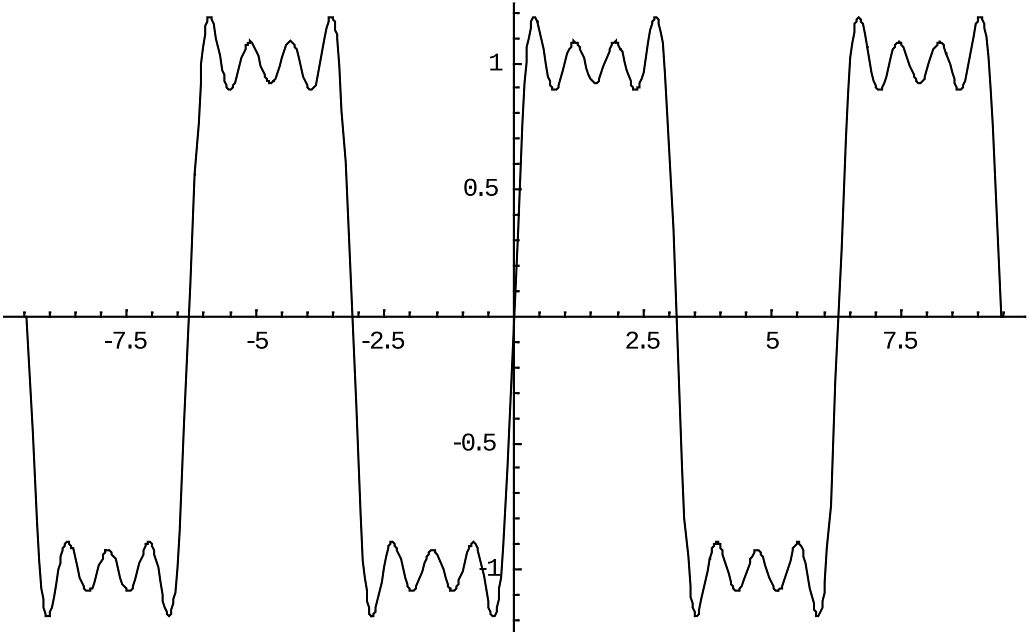
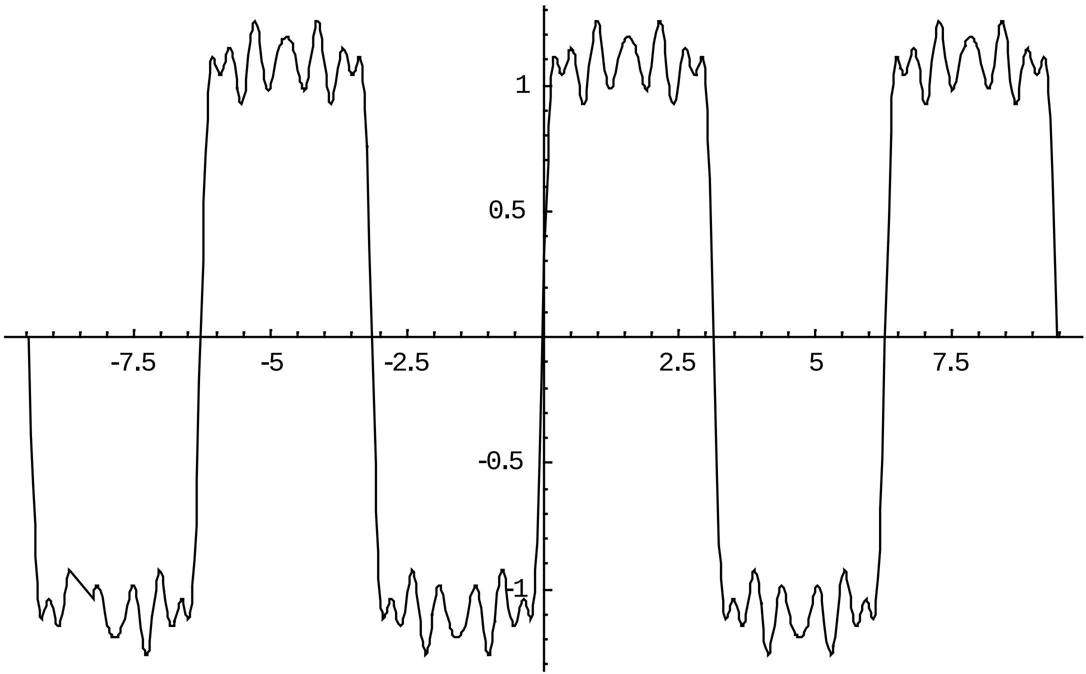


n=7



n=15