

## Laplace Transforms and Convolution

In today's lecture we define Convolution and apply it along with unit impulse response to determine the Zero State Response (ZSR) for a time-invariant  $n$ -th order linear ODE, i.e. an ODE of the form  $p(D)[x(t)] = f(t)$ . In this context we are only interested in solutions for  $t > 0$ . These methods are particularly useful for dealing with input functions  $f(t)$  defined by data rather than simply by familiar elementary functions.

### Unit Impulse Response, Weight Function, Transfer Function

Unit impulse response refers to the solution of the ODE  $p(D)[x(t)] = \delta(t)$  with rest initial conditions. The solution is also known as the *weight function* for the given differential operator  $p(D)$ . It is the simplest to tackle algebraically and we'll use it soon along with convolution to solve Initial Value Problems. We generally denote the unit impulse response (weight function) by  $w(t)$ . It's Laplace Transform  $W(s)$  is called the *transfer function*.

### Unit Step Response

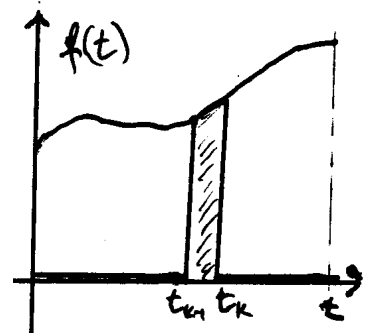
Unit step response refers to the solution of the ODE  $p(D)[x(t)] = u(t)$  with rest initial conditions. It is a bit more algebraically complicated to solve than the unit impulse response but is still relatively simple. We generally denote the unit step response by  $v(t)$ .

It is worth noting that because these differential operators are time-invariant (constant coefficients), we can use the generalized derivative to differentiate both sides of  $p(D)[x(t)] = u(t)$  to get  $D \circ p(D)[v(t)] = p(D)[D(v(t))] = p(D)[\dot{v}(t)] = D[u(t)] = \delta(t)$ , so  $p(D)[\dot{v}(t)] = \delta(t)$ . Therefore  $\dot{v}(t) = w(t)$ .

## Convolution

**Situation:** We need to solve a differential equation of the form  $p(D)[x(t)] = f(t)$  with initial conditions  $x(0) = x_0$ ,  $\dot{x}(0) = \dot{x}_0$ , etc.

**Plan:** If we can find the unit impulse response for this system (with rest initial conditions), i.e. the solution to  $p(D)[x(t)] = \delta(t)$  with initial conditions  $x(0) = 0$ ,  $\dot{x}(0) = 0$ , etc., we will develop a method for finding a solution to  $p(D)[x(t)] = f(t)$  by thinking of  $f(t)$  as a “*train of impulses.*” We will likely use Laplace transform methods to find the unit impulse response. We’ll use Riemann Sums ideas to construct an integral by piecing together solutions associated with the impulses. This will be the **convolution** integral.



## Developing the convolution integral

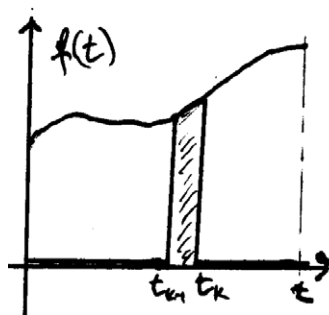
- (1) We start by solving for the unit impulse function  $w(t)$ , i.e. the solution to  $p(D)[x(t)] = \delta(t)$  with initial conditions  $x(0) = 0$ ,  $\dot{x}(0) = 0$ , etc. We refer to this by  $w(t)$  and reserve  $x(t)$  for the solution to  $p(D)[x(t)] = f(t)$ . If  $W(s)$  is the Laplace transform of  $w(t)$ , we'll have  $p(s)W(s) = 1$  where  $p(s)$  is the characteristic polynomial, so  $W(s) = \frac{1}{p(s)}$  and with some partial fractions calculations and the inverse Laplace transform, finding  $w(t)$  can be reduced to a relatively simple routine. We call  $w(t)$  the **weight function**, and we call  $W(s)$  the **transfer function**.
- (2) We use **time invariance** to declare that the translated unit impulse response for  $p(D)[x(t)] = \delta(t - t_k)$  will be  $w(t - t_k)$ . We'll use this in the integral to follow.
- (3) If we're interested in understanding what's happening during the time interval  $[0, t]$ , we start by partitioning this interval into many small subintervals  $[t_{k-1}, t_k]$ . On each of these subintervals, we can use a box function to "switch on" just one small section of the function  $f(t)$ . That is, if we let  $f_k(t) = f(t)[u(t - t_{k-1}) - u(t - t_k)]$  then this

function will be identically zero except in the  $k$ -th subinterval  $[t_{k-1}, t_k]$ . We can later reassemble the function as  $f(t) = \sum f_k(t)$ .

If  $f(t)$  is reasonably well behaved (except, perhaps, at finitely many points), we can say that within a given subinterval,

$$f_k(t) \cong f(t_k) = \int_0^{+\infty} f(t)\delta(t-t_k)dt = \int_{-\infty}^{+\infty} f(t)\delta(t-t_k)dt,$$

where we use the fact that evaluation of a function at a point is accomplished by integrating against a delta function concentrated at that point.



- (4) If the solution to  $p(D)[x(t)] = \delta(t-t_k)$  is  $w(t-t_k)$ , then linearity gives that the solution to  $p(D)[x(t)] = f(t_k)\delta(t-t_k)\Delta t_k$  will be  $f(t_k)w(t-t_k)\Delta t_k$  where  $\Delta t_k = t_k - t_{k-1}$  is the width of the  $k$ -th subinterval.  
 [We simply multiplied both sides by  $f(t_k)\Delta t_k$ .]

(5) By linearity (superposition), we can sum to get that the solution to

$$p(D)[x(t)] \cong \sum_k f(t_k)\delta(t-t_k)\Delta t_k \text{ must therefore be}$$

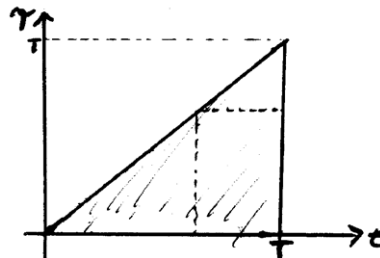
$$x(t) \cong \sum_k f(t_k)w(t-t_k)\Delta t_k = \sum_k f(t_k)w(t-\tau_k)\Delta \tau_k ,$$

where we changed to the variable  $\tau$  in anticipation of the next step.

(6) If we pass to the limit as the norm of the partition goes to zero, the sum will become an integral and the approximation will become exact, i.e.

$$x(t) = \int_0^t f(\tau)w(t-\tau)d\tau \equiv (f * w)(t) , \text{ the}$$

convolution integral. This provides a solution to  $p(D)[x(t)] = f(t)$  for the zero state (rest conditions), and we refer to this solution as the **zero state response (ZSR)**.



**Definition (Convolution):** Given two functions  $w(t)$  and  $f(t)$ , we define:

$$(f * w)(t) = \int_0^t f(\tau)w(t-\tau)d\tau .$$

It's a straightforward exercise to show that the convolution product is commutative, i.e.  $f * w = w * f$ .

## Even More Calculations

17) **Convolution:**  $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t g(\tau)f(t-\tau)d\tau = (g * f)(t)$ , i.e.

convolution is commutative.

This follows by substitution. If we let  $u = t - \tau$  ( $u$  and  $\tau$  are the variables here), then  $du = -d\tau$  and when we also adjust the integral limits we get:

$$(f * g)(t) = \int_{\tau=0}^{\tau=t} f(\tau)g(t-\tau)d\tau = -\int_{u=t}^{u=0} f(t-u)g(u)du = \int_{u=0}^{u=t} g(u)f(t-u)du = (g * f)(t)$$

18) If  $\mathcal{L}[f(t)] = F(s)$  and  $\mathcal{L}[g(t)] = G(s)$ , then  $\mathcal{L}[(f * g)(t)] = F(s)G(s)$

$$\mathcal{L}[(f * g)(t)] = \int_{0^-}^{\infty} e^{-st} \left[ \int_0^t f(\tau)g(t-\tau)d\tau \right] dt = F(s)G(s)$$

The fact that the solution to the differential equation  $p(D)x = f(t)$  will have the solution  $(f * w)(t)$  is also known as **Green's Formula**.

**Note:** When applying the convolution method to solving  $p(D)x = f(t)$  for more general initial conditions, the solution will be  $x(t) = \mathbf{ZIR} + \mathbf{ZSR}$ , where **ZIR** is the **zero input response** and **ZSR** is the **zero state response**.

The usefulness of this transform method is built on the fact that we can relatively easily find the Laplace transform for most everything that appears in a given differential equation of the form  $p(D)x = f(t)$  including the case where  $f(t)$  is given by raw data, and once we have a table of these transforms we can generally invert the process by inspection. Another essential fact is that the Laplace transform acts linearly, and this allows us to decompose complex problems into a sums of simple problems.

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**Example 4 (with a familiar input):** Solve  $\ddot{x} + 3\dot{x} + 2x = 2e^{-t}$ ,  $x(0) = 0$ ,  $\dot{x}(0) = 0$  using (a) previous methods, (b) using only the Laplace transform, and (c) using the Laplace transform plus convolution.

**Solution: (a)** For the homogeneous equation  $\ddot{x} + 3\dot{x} + 2x = 0$ , the characteristic polynomial is  $p(s) = s^2 + 3s + 2 = (s + 2)(s + 1)$  which yields the two roots  $s = -2$  and  $s = -1$ . This gives the two independent solutions  $e^{-2t}$  and  $e^{-t}$ , and all homogeneous solutions are of the form  $x_h(t) = c_1e^{-2t} + c_2e^{-t}$ .

In seeking a particular solution  $x_p(t)$  that satisfies the inhomogeneous differential equation, we see that the Exponential Response Formula (ERF) won't work – there is resonance. We can, however, use the Resonant Response Formula to get the particular

solution  $x_p(t) = \frac{2te^{-t}}{p'(-1)} = \frac{2te^{-t}}{1} = 2te^{-t}$ , so the general solution is

$x(t) = x_h(t) + x_p(t) = c_1e^{-2t} + c_2e^{-t} + 2te^{-t}$ . Its derivative is

$\dot{x}(t) = -2c_1e^{-2t} - c_2e^{-t} - 2te^{-t} + 2e^{-t}$ . Substituting the (rest) initial conditions gives

$\left. \begin{cases} x(0) = c_1 + c_2 = 0 \\ \dot{x}(0) = -2c_1 - c_2 + 2 = 0 \end{cases} \right\}$ , and these can be solved to give  $c_1 = 2, c_2 = -2$ , so the solution

is  $\boxed{x(t) = 2e^{-2t} - 2e^{-t} + 2te^{-t}}$ .

**(b)** We need the following Laplace transforms:

(1)  $\mathcal{L}(e^{kt}) = \frac{1}{s-k}$  with region of convergence  $\text{Re}(s) > k$ , so  $\mathcal{L}(e^{-2t}) = \frac{1}{s+2}$ .

(2) If the Laplace transform of  $x(t)$  is  $X(s)$ , then the Laplace transforms of its derivatives are  $\mathcal{L}(\dot{x}(t)) = sX(s) - x(0-)$  and  $\mathcal{L}(\ddot{x}(t)) = s^2X(s) - sx(0-) - \dot{x}(0-)$ .

We have rest initial conditions, so these are greatly simplified and, in fact,

$\mathcal{L}(p(D)x) = p(s)X(s)$ . Specifically,

$$\mathcal{L}(\ddot{x} + 3\dot{x} + 2x) = s^2X(s) + 3sX(s) + 2X(s) = (s^2 + 3s + 2)X(s) = p(s)X(s).$$



If we now transform the entire differential equation, we get  $(s^2 + 3s + 2)X(s) = \frac{2}{s+1}$ .

We then solve for  $X(s) = \frac{2}{(s+1)(s^2 + 3s + 2)} = \frac{2}{(s+2)(s+1)^2} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$ .

There are many good ways to find the unknowns  $A$ ,  $B$ , and  $C$ . For example, if we multiply through by the common denominator to clear fractions, we get

$2 = A(s+1)^2 + B(s+1)(s+2) + C(s+2)$ . Plugging in the specific values  $s = -2$  and  $s = -1$  quickly yields that  $A = 2$  and  $C = 2$ . Plugging in, for example,  $s = 0$  and using the values for  $A$  and  $C$  then yields  $B = -2$ . So  $X(s) = \frac{2}{s+2} - \frac{2}{s+1} + \frac{2}{(s+1)^2}$ .

Consulting our table of common Laplace transforms, we see that  $\frac{2}{s+2} = \mathcal{L}(2e^{-2t})$ ,

$\frac{2}{s+1} = \mathcal{L}(2e^{-t})$ , and  $\frac{2}{(s+1)^2} = \mathcal{L}(2te^{-t})$ , so transforming back gives

$$\boxed{x(t) = 2e^{-2t} - 2e^{-t} + 2te^{-t}}.$$

(c) We start by finding the unit impulse response, the solution to  $\ddot{x} + 3\dot{x} + 2x = \delta(t)$  with rest initial conditions  $x(0) = 0, \dot{x}(0) = 0$ . Laplace transform gives  $p(s)W(s) = 1$ , so

$$W(s) = \frac{1}{(s+2)(s+1)} = -\frac{1}{s+2} + \frac{1}{s+1}. \text{ Consulting the Laplace transform table, this yields}$$

$$\text{the weight function } w(t) = \begin{cases} 0 & t < 0 \\ -e^{-2t} + e^{-t} & t > 0 \end{cases} = u(t)(-e^{-2t} + e^{-t}).$$

With  $f(t) = 2e^{-t}$ , then convolution of the weight function and the given input signal gives:

$$\begin{aligned} (w * f)(t) &= \int_{\tau=0}^{\tau=t} w(\tau) f(t-\tau) d\tau = \int_{\tau=0}^{\tau=t} (-e^{-2\tau} + e^{-\tau}) 2e^{-(t-\tau)} d\tau \\ &= \int_{\tau=0}^{\tau=t} 2e^{-t} (-e^{-\tau} + 1) d\tau = 2e^{-t} \left[ e^{-\tau} + \tau \right]_{\tau=0}^{\tau=t} = 2e^{-t} [e^{-t} - 1 + t] \\ &= \boxed{+2e^{-2t} - 2e^{-t} + 2te^{-t} = x(t)} \end{aligned}$$

Alternatively, we could have calculated this as:

$$\begin{aligned}
 (f * w)(t) &= \int_{\tau=0}^{\tau=t} f(\tau)w(t-\tau)d\tau = \int_{\tau=0}^{\tau=t} 2e^{-\tau} (-e^{-2(t-\tau)} + e^{-(t-\tau)})d\tau \\
 &= \int_{\tau=0}^{\tau=t} (-2e^{-2t}e^{\tau} + 2e^{-t})d\tau = -2e^{-2t}(e^t - 1) + 2e^{-t}(t - 0) = \boxed{+2e^{-2t} - 2e^{-t} + 2te^{-t} = x(t)}
 \end{aligned}$$

**Example 5:** Solve the same ODE as above but with non-rest initial conditions:

$$\ddot{x} + 3\dot{x} + 2x = 2e^{-t}, \quad x(0) = 4, \quad \dot{x}(0) = 0$$

**Solution:** All of the previous steps are the same in deriving the **zero state response (ZSR)**, so we have **ZSR** =  $+2e^{-2t} - 2e^{-t} + 2te^{-t}$ .

We need only find the **zero input response (ZIR)**. This simply means that we solve  $\ddot{x} + 3\dot{x} + 2x = 0$  to get  $x_h(t) = c_1e^{-2t} + c_2e^{-t}$  and  $\dot{x}_h(t) = -2c_1e^{-2t} - c_2e^{-t}$ , so

$$\begin{cases} x_h(0) = c_1 + c_2 = 4 \\ \dot{x}_h(0) = -2c_1 - c_2 = 0 \end{cases} \Rightarrow c_1 = -4, \quad c_2 = 8.$$

So **ZIR** =  $-4e^{-2t} + 8e^{-t}$ .

Therefore  $x(t) = \mathbf{ZSR} + \mathbf{ZIR} = +2e^{-2t} - 2e^{-t} + 2te^{-t} - 4e^{-2t} + 8e^{-t} = \boxed{-2e^{-2t} + 6e^{-t} + 2te^{-t}}$ .