V6. Multiply-connected
Regions; Topology

In Section V5, we called a region D of the plane *simply-connected* if it had no holes in it. This is a typical example of what would be called in mathematics a *topological property*, that is, a property that can be described without using measurement. For a curve, such properties as having length 3, or being a circle, or a line, or a triangle — these are not topological properties, since they involve measurement, whereas the property of being closed, or of intersecting itself once, would be topological.

The important thing about topological properties is that they are preserved when the geometric figure is deformed continuously without adding or subtracting points, whereas non-topological properties change under such a deformation. For example, if you deform a circle, it will not stay a circle, but it will still remain a closed curve that does not intersect itself.

Topology is that branch of mathematics which studies topological properties of geometrical figures; it's a kind of geometry, but at the opposite pole from Euclidean geometry, which emphasizes measurement ("congruent triangles" "right angles", "circles"). Topology is a large and active branch of mathematics today, one which is attracting attention from other disciplines, like theoretical physics and molecular biology. Most students have never heard of it, because topological properties don't enter very often into the first few years of mathematics. However, they do right here, and in fact it was just in the study of the possible values of a line integral around a closed curve that the central ideas of modern topology first entered into mathematics, in the middle of the 1800's.

So let $\mathbf{F}$ be a continuously differentiable vector field in a multiply-connected — i.e., not simply-connected — region D of the xy-plane, and suppose curl $\mathbf{F} = 0$. What values can $\oint_C \mathbf{F} \cdot d\mathbf{r}$ have?

We begin by considering an earlier example (Section V2, Example 2) in greater detail, because it gives the key to the general case. Consider the vector field representing the electromagnetic field of a wire along the z-axis carrying a constant current:

$$(1) \qquad \mathbf{F} = \frac{-y\,\mathbf{i} + x\,\mathbf{j}}{r^2} \; ;$$

we know that curl $\mathbf{F} = 0$ (see the cited Example.)

To evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C does not pass through the origin, we use polar coordinates and t as parameter:

$$(2) \qquad x = r\cos\theta, \qquad y = r\sin\theta, \qquad r = r(t), \qquad \theta = \theta(t) \; .$$

As you move around C, the polar angle θ must vary continuously; thus we allow it to take on all values, and do not restrict it to lie in the interval $[0, 2\pi]$. Using (2) and the chain rule for several variables, we get

$$
\begin{aligned}
dx &= x_r dr + x_\theta d\theta = \cos\theta\, dr - r\sin\theta\, d\theta \\
dy &= \sin\theta\, dr + r\cos\theta\, d\theta \; ;
\end{aligned}
$$

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We calculate the line integral as usual; everything cancels, and we get

(3) $\qquad \oint_C \mathbf{F} \cdot d\mathbf{r} \; = \; \oint_C \dfrac{-y\,dx + x\,dy}{r^2} \; = \; \oint_C d\theta \; = \; \int_{t_0}^{t_1} \dfrac{d\theta}{dt}\,dt \; = \; \theta(t_1) - \theta(t_0) \; .$

(4) $\qquad\qquad\qquad = \; 2\pi n,$

the net change in polar angle θ as you move around C — since C is closed, this must be an integer multiple n of 2π. It is called the **winding number** of C around the origin.

By way of illustration, the value of the winding number n is given for a few closed curves C below; the origin is indicated by a dot, and it's understood that as t runs from t_0 to t_1, the curve C is traced out just once, in the direction shown.
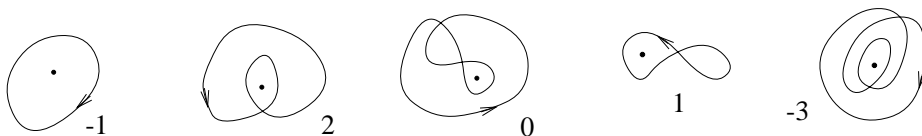

Intuitively, the winding number is the total number of times that C goes around the origin, counting $+1$ each time it goes around counterclockwise, and -1 when it goes around clockwise. The winding number of C around any other point not on C is defined the same way, by taking the point to be the origin of a polar coordinate system.

The winding number about the origin (or around any given point) is a topological property of the oriented curve C, since if C is deformed continuously without ever crossing the point, the winding numbers must also vary continuously, but the only way an integer can vary continuously is to always stay the same.

Here is a simple way of finding the winding number of C around the origin: the winding number is the total number of times that C crosses the positive x-axis, counting $+1$ each time C crosses from below, and -1 each time it crosses from above.

Instead of the positive x-axis, you can use any directed ray from the origin, counting $+1$ when C crosses the ray in the countercloswise direction, and -1 when it crosses in the clockwise direction.

Example 1.   For the curve shown, and the three choices of ray given, calculate the winding number using each in turn, and find $\oint_C \mathbf{F} \cdot d\mathbf{r}$.
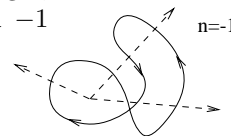
Solution.   Each ray gives $n = -1$ as the winding number. Thus $\oint_C \mathbf{F} \cdot d\mathbf{r} \; = \; -2\pi$, according to (4) above.

So far, we have shown by explicit calculation for the particular field $\mathbf{F}$ given by (1), the value of $\oint_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the winding number of C around the origin. We now consider the general situation.

Let $\mathbf{F}$ be a vector field which is continuously differentiable in a region D, and assume that curl $\mathbf{F} = 0$ in D. We will show now that the value of $\oint_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the topological properties of C, and not on its exact position or length.

To see this, we will assume that D consists of one connected region having k holes. The holes come from removing portions of the region — we might remove a point, a line segment, the interior of a circle, the letter Y, etc. We could not remove a circle, however, since then what was left would not be one connected piece.

Draw a simple closed curve C_1, directed counterclockwise, around the i-th hole, and let

$$A_i = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r}.$$

Then for any closed curve C in the region, we claim that

(6) $$\oint_C \mathbf{F} \cdot d\mathbf{r} = n_1 A_1 + \ldots + n_k A_k, \qquad n_i \text{ integers,}$$
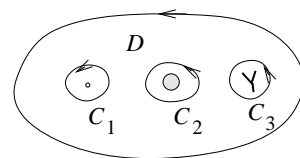
where n_i is the winding number of C around the i-th hole (i.e., the winding number of C around any point inside that i-th hole).

We shall indicate how the argument goes in a few cases; the general case would take us farther into topology than we are able to go at present. The essential tool is Green's theorem.

Example 2. Suppose C is a path like the one pictured ($k = 3$ in the picture, but this isn't significant.) Let C'_i be the curve C_i with its direction reversed. Then $\mathbf{F}$ is continuously differentiable in the region between C and the C'_i, and also curl $\mathbf{F} = 0$. Therefore by the extended form of Green's theorem (Section V5), we get

$$\oint_C \mathbf{F} \cdot d\mathbf{r} + \oint_{C'_1} \mathbf{F} \cdot d\mathbf{r} + \ldots + \oint_{C'_k} \mathbf{F} \cdot d\mathbf{r} = 0, \qquad \text{so that}$$
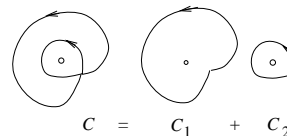
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = A_1 + \ldots + A_k,$$
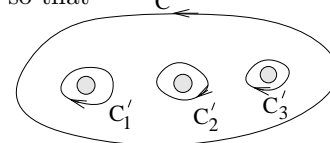
proving (6) in this case.

Example 3. If there is just one hole, Example 2 shows that for any simple closed curve going counterclockwise around the hole,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = A_1 \ .$$

(In the first example (4) discussed, $A_1 = 2\pi$.)

Suppose now that C is a closed curve going around twice, as in the illustration. Break C into the sum of two simple closed curves as shown, using a point where it crosses itself; then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 2A_1.$$

Example 4. This example combines the two above. If C is as shown (here $k = 2$), then $C = D_1 + D_2$; let D'_i be D_i with the direction reversed. Then $D_1 + D'_2$ is a simple curve having only hole number 1 in its interior. Therefore

$$\oint_{D_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{D'_2} \mathbf{F} \cdot d\mathbf{r} = A_1, \qquad \oint_{D_2} \mathbf{F} \cdot d\mathbf{r} = A_2, \qquad \text{so that} \qquad \oint_C \mathbf{F} \cdot d\mathbf{r} = A_1 + 2A_2.$$

One final remark. If $\mathbf{F}$ is as above, with curl $\mathbf{F} = 0$, and we try to find a function $f(x, y)$ for which $\mathbf{F} = \Delta f$, by defining

(7) $$f(x, y) = \int_{(a,b)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r} \ ,$$
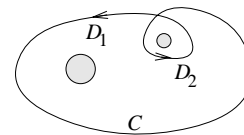
this won't work because the value of $f(x, y)$ depends not just on (x, y) but also on the path of integration used. However, equation (6) shows that two different values for the line integfral always differ by a number of the form $n_1 A_1 + \ldots + n_k A_k$, because if C_1 and C_2 are two different choices of path from (a, b) to (x, y), then $C_1 + C_2'$ is a closed path, and thus

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \;+\; \int_{C_2'} \mathbf{F} \cdot d\mathbf{r} \;=\; n_1 A_1 + \ldots + n_k A_k \;.$$

Here n_i is the winding number of the path $C_1 + C_2'$ around the i-th hole. Therefore
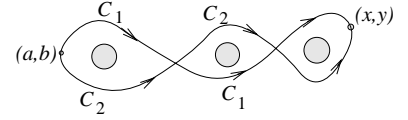
(8) $$\oint_C \mathbf{F} \cdot d\mathbf{r} \;=\; \oint_C \mathbf{F} \cdot d\mathbf{r} \;+ n_1 A_1 + \ldots + n_k A_k \;.$$

Now if (x, y) is moved a little, and the paths C_1 and C_2 are moved accordingly, the numbers $n_1, \ldots, n_k$ will not change since they are winding numbers. Thus, even though the function $f(x, y)$ defined by (7) is multiple-valued, according to (8) the different determinations of its value differ by a constant, and therefore all have the same derivative. Thus we can claim that the multiple-valued function $f(x, y)$ is differentiable, and

$$\mathbf{F} \;=\; \nabla f(x, y) \;.$$

Our conclusion is that, if curl $\mathbf{F} = 0$ in a multiply-connected region, even if the field is not conservative we can still view it as the gradient field of a multiple-valued function $f(x, y)$.

For example, going back to our first significant example, equation (3) shows that we may take $f(x, y) = \theta(x, y)$, the multiple-valued polar angle. No matter which determination of θ you pick, it is still true that for some n,

$$\theta \;=\; \tan^{-1} \frac{y}{x} \;+\; 2n\pi, \qquad \text{and}$$
$$\nabla\theta \;=\; \frac{-y\,\mathbf{i} + x\,\mathbf{j}}{r^2} \;.$$

Exercises: Section 4H