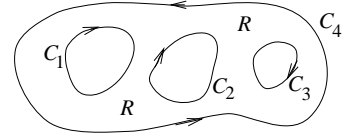


## V5. Simply-Connected Regions

### 1. The Extended Green's Theorem.

In the work on Green's theorem so far, it has been assumed that the region  $R$  has as its boundary a single simple closed curve. But this isn't necessary. Suppose the region has a boundary composed of several simple closed curves, like the ones pictured. We suppose these boundary curves  $C_1, \dots, C_m$  all lie within the domain where  $\mathbf{F}$  is continuously differentiable. Most importantly, all the curves must be directed so that the normal  $\mathbf{n}$  points *away* from  $R$ .

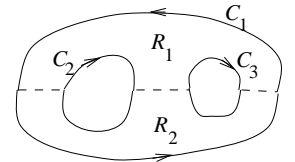


**Extended Green's Theorem** *With the curve orientations as shown,*

$$(1) \quad \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \dots + \int_{C_m} \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl } \mathbf{F} \, dA$$

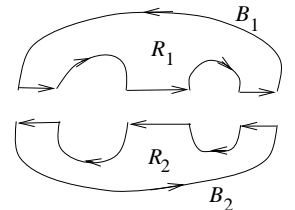
In other words, Green's theorem also applies to regions with several boundary curves, provided that we take the line integral over the complete boundary, with each part of the boundary oriented so the normal  $\mathbf{n}$  points outside  $R$ .

**Proof.** We use subdivision; the idea is adequately conveyed by an example. Consider a region with three boundary curves as shown. The three cuts illustrated divide up  $R$  into two regions  $R_1$  and  $R_2$ , each bounded by a single simple closed curve, and Green's theorem in the usual form can be applied to each piece. Letting  $B_1$  and  $B_2$  be the boundary curves shown, we have therefore



$$(2) \quad \oint_{B_1} \mathbf{F} \cdot d\mathbf{r} = \iint_{R_1} \text{curl } \mathbf{F} \, dA \quad \oint_{B_2} \mathbf{F} \cdot d\mathbf{r} = \iint_{R_2} \text{curl } \mathbf{F} \, dA$$

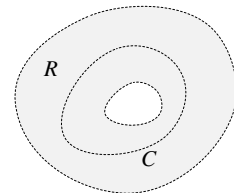
Add these two equations together. The right sides add up to the right side of (1). The left sides add up to the left side of (1) (for  $m = 2$ ), since over each of the three cuts, there are two line integrals taken in opposite directions, which therefore cancel each other out.  $\square$



### 2. Simply-connected and multiply-connected regions.

Though Green's theorem is still valid for a region with "holes" like the ones we just considered, the relation  $\text{curl } \mathbf{F} = 0 \Rightarrow \mathbf{F} = \nabla f$  is not. The reason for this is as follows.

We are trying to show that  $\text{curl } \mathbf{F} = 0 \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$  for any closed curve lying in  $R$ . We expect to be able to use Green's theorem. But if the region has a hole, like the one pictured, we cannot apply Green's theorem to the curve  $C$  because the interior of  $C$  is not entirely contained in  $R$ .



To see what a delicate affair this is, consider the earlier Example 2 in Section V2. The field  $\mathbf{G}$  there satisfies  $\text{curl } \mathbf{G} = 0$  everywhere but the origin. The region  $R$  is the  $xy$ -plane with  $(0,0)$  removed. But  $\mathbf{G}$  is not a gradient field, because  $\oint_C \mathbf{G} \cdot d\mathbf{r} \neq 0$  around a circle  $C$  surrounding the origin.

This is clearer if we use Green's theorem in normal form (Section V4). If the flow field satisfies  $\operatorname{div} \mathbf{F} = 0$  everywhere except at one point, that doesn't guarantee that the flux through every closed curve will be 0. For the spot where  $\operatorname{div} \mathbf{F}$  is undefined might be a source, through which fluid is being added to the flow.

In order to be able to prove under reasonable hypotheses that  $\operatorname{curl} \mathbf{F} = 0 \Rightarrow \mathbf{F} = \nabla f$ , we define our troubles away by assuming that  $R$  is the sort of region where the difficulties described above cannot occur—i.e., we assume that  $R$  has no holes; such regions are called *simply-connected*.

**Definition.** A two-dimensional region  $D$  of the plane consisting of one connected piece is called **simply-connected** if it has this property: whenever a simple closed curve  $C$  lies entirely in  $D$ , then its interior also lies entirely in  $D$ .

As examples: the  $xy$ -plane, the right-half plane where  $x \geq 0$ , and the unit circle with its interior are all simply-connected regions. But the  $xy$ -plane minus the origin is not simply-connected, since any circle surrounding the origin lies in  $D$ , yet its interior does not.

As indicated, one can think of a simply-connected region as one without "holes". Regions with holes are said to be *multiply-connected*, or *not simply-connected*.

**Theorem.** Let  $\mathbf{F} = M \mathbf{i} + N \mathbf{j}$  be continuously differentiable in a simply-connected region  $D$  of the  $xy$ -plane. Then in  $D$ ,

$$(3) \quad \operatorname{curl} \mathbf{F} = 0 \quad \Rightarrow \quad \mathbf{F} = \nabla f, \quad \text{for some } f(x, y); \quad \text{in terms of components,}$$

$$(3') \quad M_y = N_x \quad \Rightarrow \quad M \mathbf{i} + N \mathbf{j} = \nabla f, \quad \text{for some } f(x, y).$$

**Proof.** Since a field is a gradient field if its line integral around any closed path is 0, it suffices to show

$$(4) \quad \operatorname{curl} \mathbf{F} = 0 \quad \Rightarrow \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \quad \text{for every closed curve } C \text{ in } D.$$

We prove (4) in two steps.

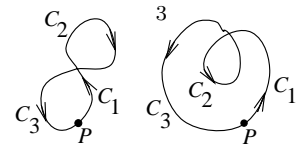
Assume first that  $C$  is a simple closed curve; let  $R$  be its interior. Then since  $D$  is simply-connected,  $R$  will lie entirely inside  $D$ . Therefore  $\mathbf{F}$  will be continuously differentiable in  $R$ , and we can use Green's theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F} \, dx dy = 0.$$

Next consider the general case, where  $C$  is closed but not simple—i.e., it intersects itself. Then  $C$  can be broken into smaller simple closed curves for which the above argument will be valid. A formal argument would be awkward to give, but the examples illustrate. In both cases, the path starts and ends at  $P$ , and

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r}.$$

In both cases,  $C_2$  is a simple closed path, and also  $C_1 + C_3$  is a simple closed path. Since  $D$  is simply-connected, the interiors automatically lie in  $D$ , so that by the first part of the argument,



$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 0 \quad \text{and} \quad \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

Adding these up, we get  $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ . □

The above argument works if  $C$  intersects itself a finite number of times. If  $C$  intersects itself infinitely often, we would have to resort to approximations to  $C$ ; we skip this case.

We pause now to summarize compactly the central result, both in the language of vector fields and in the equivalent language of differentials.

**Curl Theorem.** Let  $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$  be a continuously differentiable vector field in a simply-connected region  $D$  of the  $xy$ -plane. Then the following four statements are equivalent — if any one is true for  $\mathbf{F}$  in  $D$ , so are the other three:

$$1. \int_P^Q \mathbf{F} \cdot d\mathbf{r} \text{ is path-independent} \qquad 1.' \int_P^Q M dx + N dy \text{ is path-independent}$$

for any two points  $P, Q$  in  $D$ ;

$$2. \oint_C \mathbf{F} \cdot d\mathbf{r} = 0, \qquad 2.' \oint_C M dx + N dy = 0,$$

for any simple closed curve  $C$  lying in  $D$ ;

$$3. \mathbf{F} = \nabla f \text{ for some } f \text{ in } D \qquad 3.' M dx + N dy = df \text{ for some } f \text{ in } D$$

$$4. \text{curl } \mathbf{F} = 0 \text{ in } D \qquad 4.' M_y = N_x \text{ in } D.$$

**Remarks.** We summarize below what still holds true even if one or more of the hypotheses doesn't hold:  $D$  is not simply-connected, or the field  $\mathbf{F}$  is not differentiable everywhere in  $D$ .

1. Statements 1, 2, and 3 are equivalent even if  $\mathbf{F}$  is only continuous;  $D$  need not be simply-connected..

2. Statements 1, 2, and 3 each implies 4, if  $\mathbf{F}$  is continuously differentiable;  $D$  need not be simply-connected. (But 4 implies 1, 2, 3 only if  $D$  is simply-connected.)

**Example 1.** Is  $\mathbf{F} = xy\mathbf{i} + x^2\mathbf{j}$  a gradient field?

**Solution.** We have  $\text{curl } \mathbf{F} = x \neq 0$ , so the theorem says it is not.

**Example 2.** Is  $\frac{ydx - xdy}{y^2}$  an exact differential? If so, find all possible functions  $f(x, y)$  for which it can be written  $df$ .

**Solution.**  $M = 1/y$  and  $N = -x/y^2$  are continuously differentiable wherever  $y \neq 0$ , i.e., in the two half-planes above and below the  $x$ -axis. These are both simply-connected. In each of them,

$$M_y = -1/y^2 = N_x$$

Thus in each half-plane the differential is exact, by the theorem, and we can calculate  $f(x, y)$  by the standard methods in Section V2. They give

$$f(x, y) = \frac{x}{y} + c$$

where  $c$  is an arbitrary constant. This constant need not be the same for the two regions, since they do not touch. Thus the most general function is

$$f(x, y) = \begin{cases} x/y + c, & y > 0 \\ x/y + c', & y < 0 \end{cases}; \quad c, c' \text{ are arbitrary constants.}$$

**Example 3.** Let  $\mathbf{F} = r^n(x\mathbf{i} + y\mathbf{j})$ ,  $r = \sqrt{x^2 + y^2}$ . For which integers  $n$  is  $\mathbf{F}$  conservative? For each such, find a corresponding  $f(x, y)$  such that  $\mathbf{F} = \nabla f$ .

**Solution.** By the usual calculation, using the chain rule and the useful polar coordinate relations  $r_x = x/r$ ,  $r_y = y/r$ , we find that  $\text{curl } \mathbf{F} = 0$ . There are two cases.

**Case 1:**  $n \geq 0$ . Then  $\mathbf{F}$  is continuously differentiable in the whole  $xy$ -plane, which is simply-connected. Thus by the preceding theorem,  $\mathbf{F}$  is conservative, and we can calculate  $f(x, y)$  as in Section V2.

We use method 1 (line integration). The radial symmetry suggests using the ray  $C$  from  $(0, 0)$  to  $(x_1, y_1)$  as the path of integration, with the parametrization

$$x = x_1 t, \quad y = y_1 t, \quad 0 \leq t \leq 1;$$

also, let

$$r_1 = \sqrt{x_1^2 + y_1^2}; \quad \text{then} \quad r^n = r_1^n t^n, \quad x dx + y dy = r_1^2 t dt$$

and we get, by method 1 for finding  $f(x, y)$ ,

$$\begin{aligned} f(x_1, y_1) &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C r^n (x dx + y dy) \\ (6) \quad &= \int_0^1 r_1^{n+2} t^{n+1} dt = r_1^{n+2} \left. \frac{t^{n+2}}{n+2} \right|_0^1 = \frac{r_1^{n+2}}{n+2}. \end{aligned}$$

so that

$$(7) \quad f(x, y) = \frac{r^{n+2}}{n+2}, \quad \mathbf{F} = \nabla f, \quad n \geq 0.$$

**Case 2:**  $n < 0$ . The field  $\mathbf{F}$  is not defined at  $(0, 0)$ , so that its domain, the  $xy$ -plane with  $(0, 0)$  removed, is not simply-connected. So even though  $\text{curl } \mathbf{F} = 0$  in this region, (3) is not immediately applicable.

Nonetheless, if  $n \neq -2$ , one can check by differentiation that (7) is still valid.

If  $n = -2$ , guessing, inspection, or method 2 give  $f(x, y) = \ln r$ .

We conclude that the field in all cases is a gradient field. Note in particular that the two force fields given in section V1, representing respectively (apart from a constant factor) the fields arising from a positive charge at  $(0, 0)$  and a uniform positive charge along the  $z$ -axis, correspond to the respective cases  $n = -3$  and  $n = -2$ , and are both gradient fields:

$$\begin{aligned}\frac{x \mathbf{i} + y \mathbf{j}}{r^3} &= \nabla \left( -\frac{1}{r} \right) && (n = -3 : \text{positive charge at } (0, 0)) \\ \frac{x \mathbf{i} + y \mathbf{j}}{r^2} &= \nabla (\ln r) && (n = -2 : \text{uniform + charge on } z\text{-axis}).\end{aligned}$$

### Exercises: Section 4G