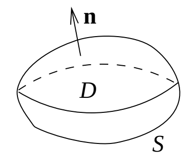
V10. The Divergence Theorem

1. Introduction; statement of the theorem.

The divergence theorem is about closed surfaces, so let's start there. By a **closed** surface S we will mean a surface consisting of one connected piece which doesn't intersect itself, and which completely encloses a single finite region D of space called its *interior*. The closed surface S is then said to be the *boundary* of D; we include S in D. A sphere, cube, and torus (an inflated bicycle inner tube) are all examples of closed surfaces. On the other hand, these are not closed surfaces: a plane, a sphere with one point removed, a tin can whose cross-section looks like a figure-8 (it intersects itself), an infinite cylinder.

A closed surface always has two sides, and it has a natural positive direction — the one for which $\mathbf{n}$ points away from the interior, i.e., points toward the outside. We shall always understand that the closed surface has been oriented this way, unless otherwise specified.



We now generalize to 3-space the normal form of Green's theorem (Section V4).

Definition. Let $\mathbf{F}(x,y,z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ be a vector field differentiable in some region D. By the **divergence** of $\mathbf{F}$ we mean the scalar function $\operatorname{div}\mathbf{F}$ of three variables defined in D by

$$\text{(1)} \qquad \operatorname{div}\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\ .$$

The divergence theorem. *Let S be a positively-oriented closed surface with interior D, and let $\mathbf{F}$ be a vector field continuously differentiable in a domain contatining D. Then*

$$\text{(2)} \qquad \iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \operatorname{div}\mathbf{F}\,dV$$

We write dV on the right side, rather than $dx\,dy\,dz$ since the triple integral is often calculated in other coordinate systems, particularly spherical coordinates. The theorem is sometimes called **Gauss' theorem**.

Physically, the divergence theorem is interpreted just like the normal form for Green's theorem. Think of $\mathbf{F}$ as a three-dimensional flow field. Look first at the left side of (2). The surface integral represents the mass transport rate across the closed surface S, with flow out of S considered as positive, flow into S as negative.

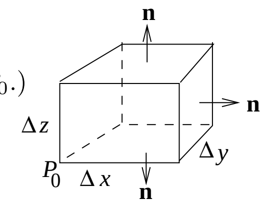
Look now at the right side of (2). In what follows, we will show that the value of $\operatorname{div}\mathbf{F}$ at (x,y,z) can be interpreted as the **source rate** at (x,y,z): the rate at which fluid is being added to the flow at this point. (Negative rate means fluid is being removed from the flow.) The integral on the right of (2) thus represents the *source rate for D*. So what the divergence theorem says is:

$$\text{(3)} \qquad \textit{flux across } S = \textit{source rate for } D\ ;$$

i.e., the net flow outward across S is the same as the rate at which fluid is being produced (or added to the flow) inside S.

To complete the argument for (3) we still have to show that

$$\text{div}\,\mathbf{F} \;=\; \textit{source rate at } (x,y,z)\ . \tag{3}$$



To see this, let $P_0 : (x_0, y_0, z_0)$ be a point inside the region D where $\mathbf{F}$ is defined. (To simplify, we denote by $(\text{div}\,\mathbf{F})_0, (\partial M/\partial x)_0$, etc., the value of these functions at P_0.)

Consider a little rectangular box, with edges $\Delta x, \Delta y, \Delta z$ parallel to the coordinate axes, and one corner at P_0. We take $\mathbf{n}$ to be always pointing outwards, as usual; thus on top of the box $\mathbf{n} = \mathbf{k}$, but on the bottom face, $\mathbf{n} = -\mathbf{k}$.

The flux across the top face in the $\mathbf{n}$ direction is approximately

$$\mathbf{F}(x_0, y_0, z_0 + \Delta z)\cdot \mathbf{k}\ \Delta x\Delta y \;=\; P(x_0, y_0, z_0 + \Delta z)\ \Delta x\Delta y,$$

while the flux across the bottom face in the $\mathbf{n}$ direction is approximately

$$\mathbf{F}(x_0, y_0, z_0)\cdot -\mathbf{k}\ \Delta x\Delta y \;=\; -P(x_0, y_0, z_0)\ \Delta x\Delta y\ .$$

So the net flux across the two faces combined is approximately

$$[P(x_0, y_0, z_0 + \Delta z) - P(x_0, y_0, z_0)]\ \Delta x\Delta y \;=\; \left(\frac{\Delta P}{\Delta z}\right)\Delta x\Delta y\Delta z\ .$$

Since the difference quotient is approximately equal to the partial derivative, we get the first line below; the reasoning for the following two lines is analogous:

$$\begin{aligned}
\textit{net flux across top and bottom} &\approx \left(\frac{\partial P}{\partial z}\right)_0 \Delta x\Delta y\Delta z;\\
\textit{net flux across two side faces} &\approx \left(\frac{\partial N}{\partial y}\right)_0 \Delta x\Delta y\Delta z;\\
\textit{net flux across front and back} &\approx \left(\frac{\partial M}{\partial x}\right)_0 \Delta x\Delta y\Delta z;
\end{aligned}$$

Adding up these three net fluxes, and using (3), we see that

$$\begin{aligned}
\textit{source rate for box} &= \textit{net flux across faces of box}\\
&\approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right)_0 \Delta x\Delta y\Delta z\ .
\end{aligned}$$

Using this, we get the interpretation for div $\mathbf{F}$ we are seeking:

$$\textit{source rate at } P_0 \;=\; \lim_{\text{box}\to 0} \frac{\textit{source rate for box}}{\textit{volume of box}} \;=\; (\text{div}\,\mathbf{F})_0\ .$$

Example 1. Verify the theorem when $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and S is the sphere $\rho = a$.

Solution. For the sphere, $\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}$; thus $\mathbf{F}\cdot\mathbf{n} = a$, and $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = 4\pi a^3$.

On the other side, div $\mathbf{F} = 3$, $\displaystyle\iiint_D 3\,dV = 3\cdot\frac{4}{3}\pi a^3$; thus the two integrals are equal. □

Example 2. Use the divergence theorem to evaluate the flux of $\mathbf{F} = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ across the sphere $\rho = a$.

Solution. Here div $\mathbf{F} = 3(x^2+y^2+z^2) = 3\rho^2$. Therefore by (2),

$$\iint_S \mathbf{F}\cdot d\mathbf{S} \;=\; 3\iiint_D \rho^2\,dV \;=\; 3\int_0^a \rho^2\cdot 4\pi\rho^2\,d\rho \;=\; \frac{12\pi a^5}{5}\ ;$$

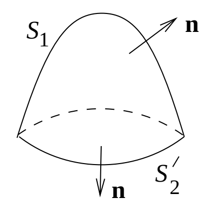
we did the triple integration by dividing up the sphere into thin concentric spheres, having volume $dV = 4\pi\rho^2\,d\rho$.

Example 3. Let S_1 be that portion of the surface of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane, and let S_2 be the part of the xy-plane lying inside the unit circle, directed so the normal $\mathbf{n}$ points upwards. Take $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$; evaluate the flux of $\mathbf{F}$ across S_1 by using the divergence theorem to relate it to the flux across S_2.

Solution. We see immediately that div $\mathbf{F} = 0$. Therefore, if we let S_2' be the same surface as S_2, but oppositely oriented (so $\mathbf{n}$ points downwards), the surface $S_1 + S_2'$ is a closed surface, with $\mathbf{n}$ pointing outwards everywhere. Hence by the divergence theorem,

$$\iint_{S_1+S_2'} \mathbf{F}\cdot d\mathbf{S} \;=\; 0 \;=\; \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} - \iint_{S_2}\mathbf{F}\cdot d\mathbf{S}$$

Therefore, since we have $\mathbf{n} = \mathbf{k}$ on S_2,



$$\begin{aligned}\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} \;=\; \iint_{S_2}\mathbf{F}\cdot d\mathbf{S} \;&=\; \iint_{S_2}\mathbf{F}\cdot\mathbf{k}\,dS \;=\; \iint_{S_2} xy\,dx\,dy\\ &=\; 0,\end{aligned}$$

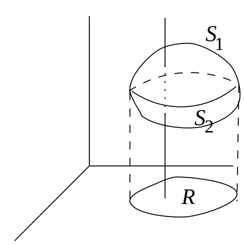
by integrating in polar coordinates (or by symmetry).

2. Proof of the divergence theorem.

We give an argument assuming first that the vector field $\mathbf{F}$ has only a $\mathbf{k}$-component: $\mathbf{F} = P(x,y,z)\,\mathbf{k}$. The theorem then says

$$(4)\qquad \iint_S P\,\mathbf{k}\cdot\mathbf{n}\,dS \;=\; \iiint_D \frac{\partial P}{\partial z}\,dV\ .$$

The closed surface S projects into a region R in the xy-plane. We assume S is vertically simple, i.e., that each vertical line over the interior of R intersects S just twice. (S can have vertical sides, however — a cylinder would be an example.) S is then described by two equations:



(5) $\quad z = g(x,y)\quad$ *(lower surface);* $\qquad z = h(x,y)\quad$ *(upper surface)*

The strategy of the proof of (4) will be to reduce each side of (4) to a double integral over R; the two double integrals will then turn out to be the same.

We do this first for the triple integral on the right of (4). Evaluating it by iteration, we get as the first step in the iteration,

$$\begin{aligned} \iiint_D \frac{\partial P}{\partial z}\, dV &= \iint_R \int_{g(x,y)}^{h(x,y)} \frac{\partial P}{\partial z}\, dz\, dx\, dy \\ &= \iint_R \big(P(x,y,h) - P(x,y,g)\big)\, dx\, dy \end{aligned} \tag{6}$$

To calculate the surface integral on the left of (4), we use the formula for the surface area element $d\mathbf{S}$ given in V9, (13):

$$d\mathbf{S} = \pm(-z_x\,\mathbf{i} - z_y\,\mathbf{j} + k)\, dx\, dy,$$

where we use the + sign if the normal vector to S has a positive k-component, i.e., points generally upwards (as on the upper surface here), and the − sign if it points generally downwards (as it does for the lower surface here).

This gives for the flux of the field $P\,\mathbf{k}$ across the upper surface S_2, on which $z = h(x, y)$,

$$\iint_{S_2} P\,\mathbf{k}\cdot d\mathbf{S} = \iint_R P(x,y,z)\, dx\, dy = \iint_R P(x,y,h(x,y))\, dx\, dy\ ,$$

while for the flux across the lower surface S_1, where $z = g(x, y)$ and we use the − sign as described above, we get

$$\iint_{S_1} P\,\mathbf{k}\cdot d\mathbf{S} = \iint_R -P(x,y,z)\, dx\, dy = \iint_R -P(x,y,g(x,y))\, dx\, dy\ ;$$

adding up the two fluxes to get the total flux across S, we have

$$\iint_S P\,\mathbf{k}\cdot d\mathbf{S} = \iint_R P(x,y,h)\, dx\, dy - \iint_R P(x,y,g)\, dx\, dy$$

which is the same as the double integral in (6). This proves (4). □

In the same way, if $\mathbf{F} = M(x, y, z)\ \mathbf{i}$ and the surface is simple in the $\mathbf{i}$ direction, we can prove

$$\iint_S M\,\mathbf{i}\cdot\mathbf{n}\, dS = \iiint_D \frac{\partial M}{\partial x}\, dV \tag{4'}$$

while if $\mathbf{F} = N(x, y, z)\ \mathbf{j}$ and the surface is simple in the $\mathbf{j}$ direction,

$$\iint_S N\,\mathbf{j}\cdot\mathbf{n}\, dS = \iiint_D \frac{\partial N}{\partial y}\, dV\ . \tag{4''}$$

Finally, for a general field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ and a closed surface S which is simple in all three directions, we have only to add up (4), (4′), and (4″). and we get the divergence theorem.

If the domain D is not bounded by a closed surface which is simple in all three directions, it can usually be divided up into smaller domains D_i which are bounded by such surfaces S_i. Adding these up gives the divergence theorem for D and S, since the surface integrals over the new faces introduced by cutting up D each occur twice, with the opposite normal vectors $\mathbf{n}$, so that they cancel out; after addition, one ends up just with the surface integral over the original S.

Exercises: Section 6C