

2. Partial Differentiation

2A. Functions and Partial Derivatives

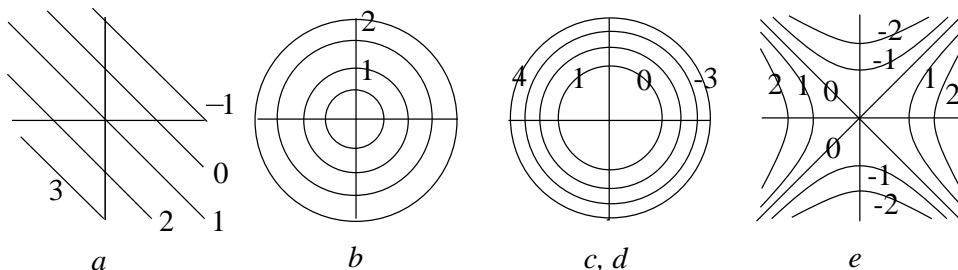
2A-1 In the pictures below, not all of the level curves are labeled. In (c) and (d), the picture is the same, but the labelings are different. In more detail:

b) the origin is the level curve 0; the other two unlabeled level curves are .5 and 1.5;

c) on the left, two level curves are labeled; the unlabeled ones are 2 and 3; the origin is the level curve 0;

d) on the right, two level curves are labeled; the unlabeled ones are -1 and -2; the origin is the level curve 1;

The crude sketches of the graph in the first octant are at the right.



2A-2 a) $f_x = 3x^2y - 3y^2$, $f_y = x^3 - 6xy + 4y$ b) $z_x = \frac{1}{y}$, $z_y = -\frac{x}{y^2}$

c) $f_x = 3\cos(3x + 2y)$, $f_y = 2\cos(3x + 2y)$

d) $f_x = 2xye^{x^2y}$, $f_y = x^2e^{x^2y}$ e) $z_x = \ln(2x + y) + \frac{2x}{2x + y}$, $z_y = \frac{x}{2x + y}$

f) $f_x = 2xz$, $f_y = -2z^3$, $f_z = x^2 - 6yz^2$

2A-3 a) both sides are $mnx^{m-1}y^{n-1}$

b) $f_x = \frac{y}{(x+y)^2}$, $f_{xy} = (f_x)_y = \frac{x-y}{(x+y)^3}$; $f_y = \frac{-x}{(x+y)^2}$, $f_{yx} = \frac{-(y-x)}{(x+y)^3}$.

c) $f_x = -2x\sin(x^2 + y)$, $f_{xy} = (f_x)_y = -2x\cos(x^2 + y)$;
 $f_y = -\sin(x^2 + y)$, $f_{yx} = -\cos(x^2 + y) \cdot 2x$.

d) both sides are $f'(x)g'(y)$.

2A-4 $(f_x)_y = ax + 6y$, $(f_y)_x = 2x + 6y$; therefore $f_{xy} = f_{yx} \Leftrightarrow a = 2$. By inspection, one sees that if $a = 2$, $f(x, y) = x^2y + 3xy^2$ is a function with the given f_x and f_y .

2A-5

a) $w_x = ae^{ax} \sin ay$, $w_{xx} = a^2e^{ax} \sin ay$;
 $w_y = e^{ax}a \cos ay$, $w_{yy} = e^{ax}a^2(-\sin ay)$; therefore $w_{yy} = -w_{xx}$.

b) We have $w_x = \frac{2x}{x^2 + y^2}$, $w_{xx} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}$. If we interchange x and y , the function $w = \ln(x^2 + y^2)$ remains the same, while w_{xx} gets turned into w_{yy} ; since the interchange just changes the sign of the right hand side, it follows that $w_{yy} = -w_{xx}$.

2B. Tangent Plane; Linear Approximation

2B-1 a) $z_x = y^2$, $z_y = 2xy$; therefore at $(1, 1, 1)$, we get $z_x = 1$, $z_y = 2$, so that the tangent plane is $z = 1 + (x - 1) + 2(y - 1)$, or $z = x + 2y - 2$.

b) $w_x = -y^2/x^2$, $w_y = 2y/x$; therefore at (1,2,4), we get $w_x = -4$, $w_y = 4$, so that the tangent plane is $w = 4 - 4(x-1) + 4(y-2)$, or $w = -4x + 4y$.

2B-2 a) $z_x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{z}$; by symmetry (interchanging x and y), $z_y = \frac{y}{z}$; then the tangent plane is $z = z_0 + \frac{x_0}{z_0}(x-x_0) + \frac{y_0}{z_0}(y-y_0)$, or $z = \frac{x_0}{z_0}x + \frac{y_0}{z_0}y$, since $x_0^2 + y_0^2 = z_0^2$.

b) The line is $x = x_0t$, $y = y_0t$, $z = z_0t$; substituting into the equations of the cone and the tangent plane, both are satisfied for all values of t ; this shows the line lies on both the cone and tangent plane (this can also be seen geometrically).

2B-3 Letting x, y, z be respectively the lengths of the two legs and the hypotenuse, we have $z = \sqrt{x^2 + y^2}$; thus the calculation of partial derivatives is the same as in **2B-2**, and we get $\Delta z \approx \frac{3}{5}\Delta x + \frac{4}{5}\Delta y$. Taking $\Delta x = \Delta y = .01$, we get $\Delta z \approx \frac{7}{5}(.01) = .014$.

2B-4 From the formula, we get $R = \frac{R_1 R_2}{R_1 + R_2}$. From this we calculate

$$\frac{\partial R}{\partial R_1} = \left(\frac{R_2}{R_1 + R_2} \right)^2, \text{ and by symmetry, } \frac{\partial R}{\partial R_2} = \left(\frac{R_1}{R_1 + R_2} \right)^2.$$

Substituting $R_1 = 1$, $R_2 = 2$ the approximation formula then gives $\Delta R = \frac{4}{9}\Delta R_1 + \frac{1}{9}\Delta R_2$.

By hypothesis, $|\Delta R_i| \leq .1$, for $i = 1, 2$, so that $|\Delta R| \leq \frac{4}{9}(.1) + \frac{1}{9}(.1) = \frac{5}{9}(.1) \approx .06$; thus

$$R = \frac{2}{3} = .67 \pm .06.$$

2B-5 a) We have $f(x, y) = (x+y+2)^2$, $f_x = 2(x+y+2)$, $f_y = 2(x+y+2)$. Therefore

at $(0, 0)$, $f_x(0, 0) = f_y(0, 0) = 4$, $f(0, 0) = 4$; linearization is $4 + 4x + 4y$;

at $(1, 2)$, $f_x(1, 2) = f_y(1, 2) = 10$, $f(1, 2) = 25$;

linearization is $10(x-1) + 10(y-2) + 25$, or $10x + 10y - 5$.

b) $f = e^x \cos y$; $f_x = e^x \cos y$; $f_y = -e^x \sin y$.

linearization at $(0, 0)$: $1 + x$; linearization at $(0, \pi/2)$: $-(y - \pi/2)$

2B-6 We have $V = \pi r^2 h$, $\frac{\partial V}{\partial r} = 2\pi r h$, $\frac{\partial V}{\partial h} = \pi r^2$; $\Delta V \approx \left(\frac{\partial V}{\partial r} \right)_0 \Delta r + \left(\frac{\partial V}{\partial h} \right)_0 \Delta h$.

Evaluating the partials at $r = 2$, $h = 3$, we get

$$\Delta V \approx 12\pi \Delta r + 4\pi \Delta h.$$

Assuming the same accuracy $|\Delta r| \leq \epsilon$, $|\Delta h| \leq \epsilon$ for both measurements, we get

$$|\Delta V| \leq 12\pi \epsilon + 4\pi \epsilon = 16\pi \epsilon, \text{ which is } < .1 \text{ if } \epsilon < \frac{1}{160\pi} < .002.$$

2B-7 We have $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} \frac{y}{x}$; $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$.

Therefore at $(3, 4)$, $r = 5$, and $\Delta r \approx \frac{3}{5}\Delta x + \frac{4}{5}\Delta y$. If $|\Delta x|$ and $|\Delta y|$ are both $\leq .01$, then

$$|\Delta r| \leq \frac{3}{5}|\Delta x| + \frac{4}{5}|\Delta y| = \frac{7}{5}(.01) = .014 \text{ (or } .02).$$

Similarly, $\frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2}$; $\frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2}$, so at the point $(3, 4)$,

$$|\Delta\theta| \leq \left| \frac{-4}{25}\Delta x \right| + \left| \frac{3}{25}\Delta y \right| \leq \frac{7}{25}(.01) = .0028 \text{ (or .003)}.$$

Since at (3, 4) we have $|r_y| > |r_x|$, r is more sensitive there to changes in y ; by analogous reasoning, θ is more sensitive there to x .

2B-9 a) $w = x^2(y+1)$; $w_x = 2x(y+1) = 2$ at (1, 0), and $w_y = x^2 = 1$ at (1, 0); therefore w is more sensitive to changes in x around this point.

b) To first order approximation, $\Delta w \approx 2\Delta x + \Delta y$, using the above values of the partial derivatives.

If we want $\Delta w = 0$, then by the above, $2\Delta x + \Delta y = 0$, or $\Delta y/\Delta x = -2$.

2C. Differentials; Approximations

$$\begin{array}{ll} \mathbf{2C-1} \text{ a) } dw = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} & \text{b) } dw = 3x^2y^2z dx + 2x^3yz dy + x^3y^2dz \\ \text{c) } dz = \frac{2y dx - 2x dy}{(x+y)^2} & \text{d) } dw = \frac{t du - u dt}{t\sqrt{t^2 - u^2}} \end{array}$$

2C-2 The volume is $V = xyz$; so $dV = yz dx + xz dy + xy dz$. For $x = 5$, $y = 10$, $z = 20$,

$$\Delta V \approx dV = 200 dx + 100 dy + 50 dz,$$

from which we see that $|\Delta V| \leq 350(.1)$; therefore $V = 1000 \pm 35$.

2C-3 a) $A = \frac{1}{2}ab \sin \theta$. Therefore, $dA = \frac{1}{2}(b \sin \theta da + a \sin \theta db + ab \cos \theta d\theta)$.

b) $dA = \frac{1}{2}(2 \cdot \frac{1}{2} da + 1 \cdot \frac{1}{2} db + 1 \cdot 2 \cdot \frac{1}{2} \sqrt{3} d\theta) = \frac{1}{2}(da + \frac{1}{2} db + \sqrt{3} d\theta)$; therefore most sensitive to θ , least sensitive to b , since $d\theta$ and db have respectively the largest and smallest coefficients.

$$\text{c) } dA = \frac{1}{2}(.02 + .01 + 1.73(.02)) \approx \frac{1}{2}(.065) \approx .03$$

$$\begin{array}{l} \mathbf{2C-4} \text{ a) } P = \frac{kT}{V}; \text{ therefore } dP = \frac{k}{V} dT - \frac{kT}{V^2} dV \\ \text{b) } V dP + P dV = k dT; \text{ therefore } dP = \frac{k dT - P dV}{V}. \\ \text{c) } \text{Substituting } P = kT/V \text{ into (b) turns it into (a).} \end{array}$$

$$\begin{array}{l} \mathbf{2C-5} \text{ a) } -\frac{dw}{w^2} = -\frac{dt}{t^2} - \frac{du}{u^2} - \frac{dv}{v^2}; \quad \text{therefore } dw = w^2 \left(\frac{dt}{t^2} + \frac{du}{u^2} + \frac{dv}{v^2} \right). \\ \text{b) } 2u du + 4v dv + 6w dw = 0; \quad \text{therefore } dw = -\frac{u du + 2v dv}{3w}. \end{array}$$

2D. Gradient; Directional Derivative

$$\begin{array}{l} \mathbf{2D-1} \text{ a) } \nabla f = 3x^2 \mathbf{i} + 6y^2 \mathbf{j}; \quad (\nabla f)_P = 3 \mathbf{i} + 6 \mathbf{j}; \quad \left. \frac{df}{ds} \right|_{\mathbf{u}} = (3 \mathbf{i} + 6 \mathbf{j}) \cdot \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}} = -\frac{3\sqrt{2}}{2} \\ \text{b) } \nabla w = \frac{y}{z} \mathbf{i} + \frac{x}{z} \mathbf{j} - \frac{xy}{z^2} \mathbf{k}; \quad (\nabla w)_P = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}; \quad \left. \frac{dw}{ds} \right|_{\mathbf{u}} = (\nabla w)_P \cdot \frac{\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}}{3} = -\frac{1}{3} \\ \text{c) } \nabla z = (\sin y - y \sin x) \mathbf{i} + (x \cos y + \cos x) \mathbf{j}; \quad (\nabla z)_P = \mathbf{i} + \mathbf{j}; \\ \left. \frac{dz}{ds} \right|_{\mathbf{u}} = (\mathbf{i} + \mathbf{j}) \cdot \frac{-3\mathbf{i} + 4\mathbf{j}}{5} = \frac{1}{5} \end{array}$$

$$\text{d) } \nabla w = \frac{2\mathbf{i} + 3\mathbf{j}}{2t + 3u}; \quad (\nabla w)_P = 2\mathbf{i} + 3\mathbf{j}; \quad \left. \frac{dw}{ds} \right|_{\mathbf{u}} = (2\mathbf{i} + 3\mathbf{j}) \cdot \frac{4\mathbf{i} - 3\mathbf{j}}{5} = -\frac{1}{5}$$

$$\text{e) } \nabla f = 2(u + 2v + 3w)(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}); \quad (\nabla f)_P = 4(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

$$\left. \frac{df}{ds} \right|_{\mathbf{u}} = 4(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot \frac{-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = -\frac{4}{3}$$

$$\mathbf{2D-2} \text{ a) } \nabla w = \frac{4\mathbf{i} - 3\mathbf{j}}{4x - 3y}; \quad (\nabla w)_P = 4\mathbf{i} - 3\mathbf{j}$$

$\left. \frac{dw}{ds} \right|_{\mathbf{u}} = (4\mathbf{i} - 3\mathbf{j}) \cdot \mathbf{u}$ has maximum 5, in the direction $\mathbf{u} = \frac{4\mathbf{i} - 3\mathbf{j}}{5}$,
and minimum -5 in the opposite direction.

$$\left. \frac{dw}{ds} \right|_{\mathbf{u}} = 0 \text{ in the directions } \pm \frac{3\mathbf{i} + 4\mathbf{j}}{5}.$$

$$\text{b) } \nabla w = \langle y + z, x + z, x + y \rangle; \quad (\nabla w)_P = \langle 1, 3, 0 \rangle;$$

$$\max \left. \frac{dw}{ds} \right|_{\mathbf{u}} = \sqrt{10}, \text{ direction } \frac{\mathbf{i} + 3\mathbf{j}}{\sqrt{10}}; \quad \min \left. \frac{dw}{ds} \right|_{\mathbf{u}} = -\sqrt{10}, \text{ direction } -\frac{\mathbf{i} + 3\mathbf{j}}{\sqrt{10}};$$

$$\left. \frac{dw}{ds} \right|_{\mathbf{u}} = 0 \text{ in the directions } \mathbf{u} = \pm \frac{-3\mathbf{i} + \mathbf{j} + c\mathbf{k}}{\sqrt{10 + c^2}} \text{ (for all } c\text{)}$$

$$\text{c) } \nabla z = 2\sin(t - u)\cos(t - u)(\mathbf{i} - \mathbf{j}) = \sin 2(t - u)(\mathbf{i} - \mathbf{j}); \quad (\nabla z)_P = \mathbf{i} - \mathbf{j};$$

$$\max \left. \frac{dz}{ds} \right|_{\mathbf{u}} = \sqrt{2}, \text{ direction } \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}; \quad \min \left. \frac{dz}{ds} \right|_{\mathbf{u}} = -\sqrt{2}, \text{ direction } -\frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}};$$

$$\left. \frac{dz}{ds} \right|_{\mathbf{u}} = 0 \text{ in the directions } \pm \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}$$

$$\mathbf{2D-3} \text{ a) } \nabla f = \langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \rangle; \quad (\nabla f)_P = \langle 4, 12, 36 \rangle; \quad \text{normal at } P: \langle 1, 3, 9 \rangle;$$

tangent plane at P : $x + 3y + 9z = 18$

$$\text{b) } \nabla f = \langle 2x, 8y, 18z \rangle; \quad \text{normal at } P: \langle 1, 4, 9 \rangle, \quad \text{tangent plane: } x + 4y + 9z = 14.$$

$$\text{c) } (\nabla w)_P = \langle 2x_0, 2y_0, -2z_0 \rangle; \quad \text{tangent plane: } x_0(x - x_0) + y_0(y - y_0) - z_0(z - z_0) = 0,$$

or $x_0x + y_0y - z_0z = 0$, since $x_0^2 + y_0^2 - z_0^2 = 0$.

$$\mathbf{2D-4} \text{ a) } \nabla T = \frac{2x\mathbf{i} + 2y\mathbf{j}}{x^2 + y^2}; \quad (\nabla T)_P = \frac{2\mathbf{i} + 4\mathbf{j}}{5};$$

T is increasing at P most rapidly in the direction of $(\nabla T)_P$, which is $\frac{\mathbf{i} + 2\mathbf{j}}{\sqrt{5}}$.

$$\text{b) } |\nabla T| = \frac{2}{\sqrt{5}} = \text{rate of increase in direction } \frac{\mathbf{i} + 2\mathbf{j}}{\sqrt{5}}. \text{ Call the distance to go } \Delta s, \text{ then}$$

$$\frac{2}{\sqrt{5}} \Delta s = .20 \quad \Rightarrow \quad \Delta s = \frac{.2\sqrt{5}}{2} = \frac{\sqrt{5}}{10} \approx .22.$$

$$\text{c) } \left. \frac{dT}{ds} \right|_{\mathbf{u}} = (\nabla T)_P \cdot \mathbf{u} = \frac{2\mathbf{i} + 4\mathbf{j}}{5} \cdot \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} = \frac{6}{5\sqrt{2}};$$

$$\frac{6}{5\sqrt{2}} \Delta s = .12 \quad \Rightarrow \quad \Delta s = \frac{5\sqrt{2}}{6} (.12) \approx (.10)(\sqrt{2}) \approx .14$$

$$\text{d) } \text{In the directions orthogonal to the gradient: } \pm \frac{2\mathbf{i} - \mathbf{j}}{\sqrt{5}}$$

2D-5 a) isotherms = the level surfaces $x^2 + 2y^2 + 2z^2 = c$, which are ellipsoids.

b) $\nabla T = \langle 2x, 4y, 4z \rangle$; $(\nabla T)_P = \langle 2, 4, 4 \rangle$; $|(\nabla T)_P| = 6$;
for most rapid decrease, use direction of $-(\nabla T)_P$: $-\frac{1}{3}\langle 1, 2, 2 \rangle$

c) let Δs be distance to go; then $-6(\Delta s) = -1.2$; $\Delta s \approx .2$

d) $\left. \frac{dT}{ds} \right|_{\mathbf{u}} = (\nabla T)_P \cdot \mathbf{u} = \langle 2, 4, 4 \rangle \cdot \frac{\langle 1, -2, 2 \rangle}{3} = \frac{2}{3}$; $\frac{2}{3}\Delta s \approx .10 \Rightarrow \Delta s \approx .15$.

2D-6 $\nabla uv = \langle (uv)_x, (uv)_y \rangle = \langle uv_x + vu_x, uv_y + vu_y \rangle = \langle uv_x, uv_y \rangle + \langle vu_x, vu_y \rangle = u\nabla v + v\nabla u$

$$\nabla(uv) = u\nabla v + v\nabla u \Rightarrow \nabla(uv) \cdot \mathbf{u} = u\nabla v \cdot \mathbf{u} + v\nabla u \cdot \mathbf{u} \Rightarrow \left. \frac{d(uv)}{ds} \right|_{\mathbf{u}} = u \left. \frac{dv}{ds} \right|_{\mathbf{u}} + v \left. \frac{du}{ds} \right|_{\mathbf{u}}.$$

2D-7 At P , let $\nabla w = a\mathbf{i} + b\mathbf{j}$. Then

$$\begin{aligned} a\mathbf{i} + b\mathbf{j} \cdot \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} &= 2 \Rightarrow a + b = 2\sqrt{2} \\ a\mathbf{i} + b\mathbf{j} \cdot \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}} &= 1 \Rightarrow a - b = \sqrt{2} \end{aligned}$$

Adding and subtracting the equations on the right, we get $a = \frac{3}{2}\sqrt{2}$, $b = \frac{1}{2}\sqrt{2}$.

2D-8 We have $P(0, 0, 0) = 32$; we wish to decrease it to 31.1 by traveling the shortest distance from the origin $\mathbf{0}$; for this we should travel in the direction of $-(\nabla P)_{\mathbf{0}}$.

$$\nabla P = \langle (y+2)e^z, (x+1)e^z, (x+1)(y+2)e^z \rangle; \quad (\nabla P)_{\mathbf{0}} = \langle 2, 1, 2 \rangle. \quad |(\nabla P)_{\mathbf{0}}| = 3.$$

Since $(-3) \cdot (\Delta s) = -.9 \Rightarrow \Delta s = .3$, we should travel a distance .3 in the direction of $-(\nabla P)_{\mathbf{0}}$. Since $|-\langle 2, 1, 2 \rangle| = 3$, the distance .3 will be $\frac{1}{10}$ of the distance from $(0, 0, 0)$ to $(-2, -1, -2)$, which will bring us to $(-.2, -.1, -.2)$.

2D-9 In these, we use $\left. \frac{dw}{ds} \right|_{\mathbf{u}} \approx \frac{\Delta w}{\Delta s}$: we travel in the direction \mathbf{u} from a given point P to the nearest level curve C ; then Δs is the distance traveled (estimate it by using the unit distance), and Δw is the corresponding change in w (estimate it by using the labels on the level curves).

a) The *direction* of ∇f is perpendicular to the level curve at A , in the increasing sense (the “uphill” direction). The *magnitude* of ∇f is the directional derivative in that direction: from the picture, $\frac{\Delta w}{\Delta s} \approx \frac{1}{.5} = 2$.

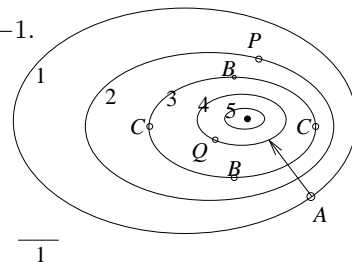
b), c) $\frac{\partial w}{\partial x} = \left. \frac{dw}{ds} \right|_{\mathbf{i}}$, $\frac{\partial w}{\partial y} = \left. \frac{dw}{ds} \right|_{\mathbf{j}}$, so B will be where \mathbf{i} is tangent to the level curve and C where \mathbf{j} is tangent to the level curve.

d) At P , $\frac{\partial w}{\partial x} = \left. \frac{dw}{ds} \right|_{\mathbf{i}} \approx \frac{\Delta w}{\Delta s} \approx \frac{-1}{5/3} = -.6$; $\frac{\partial w}{\partial y} = \left. \frac{dw}{ds} \right|_{\mathbf{j}} \approx \frac{\Delta w}{\Delta s} \approx \frac{-1}{1} = -1$.

e) If \mathbf{u} is the direction of $\mathbf{i} + \mathbf{j}$, we have $\left. \frac{dw}{ds} \right|_{\mathbf{u}} \approx \frac{\Delta w}{\Delta s} \approx \frac{1}{.5} = 2$

f) If \mathbf{u} is the direction of $\mathbf{i} - \mathbf{j}$, we have $\left. \frac{dw}{ds} \right|_{\mathbf{u}} \approx \frac{\Delta w}{\Delta s} \approx \frac{-1}{5/4} = -.8$

g) The gradient is 0 at a local extremum point: here at the point marked giving the location of the hilltop.



2E. Chain Rule**2E-1**

a) (i) $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = yz \cdot 1 + xz \cdot 2t + xy \cdot 3t^2 = t^5 + 2t^5 + 3t^5 = 6t^5$
(ii) $w = xyz = t^6$; $\frac{dw}{dt} = 6t^5$

b) (i) $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = 2x(-\sin t) - 2y(\cos t) = -4 \sin t \cos t$
(ii) $w = x^2 - y^2 = \cos^2 t - \sin^2 t = \cos 2t$; $\frac{dw}{dt} = -2 \sin 2t$

c) (i) $\frac{dw}{dt} = \frac{2u}{u^2 + v^2}(-2 \sin t) + \frac{2v}{u^2 + v^2}(2 \cos t) = -\cos t \sin t + \sin t \cos t = 0$
(ii) $w = \ln(u^2 + v^2) = \ln(4 \cos^2 t + 4 \sin^2 t) = \ln 4$; $\frac{dw}{dt} = 0$.

2E-2 a) The value $t = 0$ corresponds to the point $(x(0), y(0)) = (1, 0) = P$.

$$\left. \frac{dw}{dt} \right|_0 = \left. \frac{\partial w}{\partial x} \right|_P \left. \frac{dx}{dt} \right|_0 + \left. \frac{\partial w}{\partial y} \right|_P \left. \frac{dy}{dt} \right|_0 = -2 \sin t + 3 \cos t \Big|_0 = 3.$$

b) $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = y(-\sin t) + x(\cos t) = -\sin^2 t + \cos^2 t = \cos 2t$.

$$\frac{dw}{dt} = 0 \text{ when } 2t = \frac{\pi}{2} + n\pi, \text{ therefore when } t = \frac{\pi}{4} + \frac{n\pi}{2}.$$

c) $t = 1$ corresponds to the point $(x(1), y(1), z(1)) = (1, 1, 1)$.

$$\left. \frac{df}{dt} \right|_1 = 1 \cdot \left. \frac{dx}{dt} \right|_1 - 1 \cdot \left. \frac{dy}{dt} \right|_1 + 2 \cdot \left. \frac{dz}{dt} \right|_1 = 1 \cdot 1 - 1 \cdot 2 + 2 \cdot 3 = 5.$$

d) $\frac{df}{dt} = 3x^2 y \frac{dx}{dt} + (x^3 + z) \frac{dy}{dt} + y \frac{dz}{dt} = 3t^4 \cdot 1 + 2x^3 \cdot 2t + t^2 \cdot 3t^2 = 10t^4$.

2E-3 a) Let $w = uv$, where $u = u(t)$, $v = v(t)$; $\frac{dw}{dt} = \frac{\partial w}{\partial u} \frac{du}{dt} + \frac{\partial w}{\partial v} \frac{dv}{dt} = v \frac{du}{dt} + u \frac{dv}{dt}$.

b) $\frac{d(uvw)}{dt} = vw \frac{du}{dt} + uw \frac{dv}{dt} + uv \frac{dw}{dt}$; $e^{2t} \sin t + 2te^{2t} \sin t + te^{2t} \cos t$

2E-4 The values $u = 1$, $v = 1$ correspond to the point $x = 0$, $y = 1$. At this point,

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} = 2 \cdot 2u + 3 \cdot v = 2 \cdot 2 + 3 = 7.$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} = 2 \cdot (-2v) + 3 \cdot u = 2 \cdot (-2) + 3 \cdot 1 = -1.$$

2E-5 a) $w_r = w_x x_r + w_y y_r = w_x \cos \theta + w_y \sin \theta$

$$w_\theta = w_x x_\theta + w_y y_\theta = w_x(-r \sin \theta) + w_y(r \cos \theta)$$

Therefore,

$$\begin{aligned} (w_r)^2 + (w_\theta/r)^2 &= (w_x)^2(\cos^2 \theta + \sin^2 \theta) + (w_y)^2(\sin^2 \theta + \cos^2 \theta) + 2w_x w_y \cos \theta \sin \theta - 2w_x w_y \sin \theta \cos \theta \\ &= (w_x)^2 + (w_y)^2. \end{aligned}$$

b) The point $r = \sqrt{2}$, $\theta = \pi/4$ in polar coordinates corresponds in rectangular coordinates to the point $x = 1$, $y = 1$. Using the chain rule equations in part (a),

$$w_r = w_x \cos \theta + w_y \sin \theta; \quad w_\theta = w_x(-r \sin \theta) + w_y(r \cos \theta)$$

but evaluating all the partial derivatives at the point, we get

$$w_r = 2 \cdot \frac{1}{2}\sqrt{2} - 1 \cdot \frac{1}{2}\sqrt{2} = \frac{1}{2}\sqrt{2}; \quad \frac{w_\theta}{r} = 2(-\frac{1}{2})\sqrt{2} - \frac{1}{2}\sqrt{2} = -\frac{3}{2}\sqrt{2};$$

$$(w_r)^2 + \frac{1}{r}(w_\theta)^2 = \frac{1}{2} + \frac{9}{2} = 5; \quad (w_x)^2 + (w_y)^2 = 2^2 + (-1)^2 = 5.$$

2E-6 $w_u = w_x \cdot 2u + w_y \cdot 2v$; $w_v = w_x \cdot (-2v) + w_y \cdot 2u$, by the chain rule. Therefore

$$(w_u)^2 + (w_v)^2 = [4u^2(w_x) + 4v^2(w_y)^2 + 4uvw_xw_y] + [4v^2(w_x) + 4u^2(w_y)^2 - 4uvw_xw_y]$$

$$= 4(u^2 + v^2)[(w_x)^2 + (w_y)^2].$$

2E-7 By the chain rule, $f_u = f_x x_u + f_y y_u$, $f_v = f_x x_v + f_y y_v$; therefore

$$\langle f_u \ f_v \rangle = \langle f_x \ f_y \rangle \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

2E-8 a) By the chain rule for functions of one variable,

$$\frac{\partial w}{\partial x} = f'(u) \cdot \frac{\partial u}{\partial x} = f'(u) \cdot -\frac{y}{x^2}; \quad \frac{\partial w}{\partial y} = f'(u) \cdot \frac{\partial u}{\partial y} = f'(u) \cdot \frac{1}{x};$$

Therefore,

$$x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} = f'(u) \cdot -\frac{y}{x} + f'(u) \cdot \frac{y}{x} = 0.$$

2F. Maximum-minimum Problems

2F-1 In these, denote by $D = x^2 + y^2 + z^2$ the square of the distance from the point (x, y, z) to the origin; then the point which minimizes D will also minimize the actual distance.

a) Since $z^2 = \frac{1}{xy}$, we get on substituting, $D = x^2 + y^2 + \frac{1}{xy}$. with x and y independent; setting the partial derivatives equal to zero, we get

$$D_x = 2x - \frac{1}{x^2 y} = 0; \quad D_y = 2y - \frac{1}{y^2 x} = 0; \quad \text{or} \quad 2x^2 = \frac{1}{xy}, \quad 2y^2 = \frac{1}{xy}.$$

Solving, we see first that $x^2 = \frac{1}{2xy} = y^2$, from which $y = \pm x$.

If $y = x$, then $x^4 = \frac{1}{2}$ and $x = y = 2^{-1/4}$, and so $z = 2^{1/4}$; if $y = -x$, then $x^4 = -\frac{1}{2}$ and there are no solutions. Thus the unique point is $(1/2^{1/4}, 1/2^{1/4}, 2^{1/4})$.

b) Using the relation $x^2 = 1 + yz$ to eliminate x , we have $D = 1 + yz + y^2 + z^2$, with y and z independent; setting the partial derivatives equal to zero, we get

$$D_y = 2y + z = 0, \quad D_z = 2z + y = 0;$$

solving, these equations only have the solution $y = z = 0$; therefore $x = \pm 1$, and there are two points: $(\pm 1, 0, 0)$, both at distance 1 from the origin.

2F-2 Letting x be the length of the ends, y the length of the sides, and z the height, we have

$$\text{total area of cardboard } A = 3xy + 4xz + 2yz, \quad \text{volume } V = xyz = 1.$$

Eliminating z to make the remaining variables independent, and equating the partials to zero, we get

$$A = 3xy + \frac{4}{y} + \frac{2}{x}; \quad A_x = 3y - \frac{2}{x^2} = 0, \quad A_y = 3x - \frac{4}{y^2} = 0.$$

From these last two equations, we get

$$3xy = \frac{2}{x}, \quad 3xy = \frac{4}{y} \quad \Rightarrow \quad \frac{2}{x} = \frac{4}{y} \quad \Rightarrow \quad y = 2x$$

$$\Rightarrow \quad 3x^3 = 1 \quad \Rightarrow \quad x = \frac{1}{3^{1/3}}, \quad y = \frac{2}{3^{1/3}}, \quad z = \frac{1}{xy} = \frac{3^{2/3}}{2} = \frac{3}{2 \cdot 3^{1/3}};$$

therefore the proportions of the most economical box are $x : y : z = 1 : 2 : \frac{3}{2}$.

2F-5 The cost is $C = xy + xz + 4yz + 4xz$, where the successive terms represent in turn the bottom, back, two sides, and front; i.e., the problem is:

$$\text{minimize: } C = xy + 5xz + 4yz, \quad \text{with the constraint: } xyz = V = 2.5$$

Substituting $z = V/xy$ into C , we get

$$C = xy + \frac{5V}{y} + \frac{4V}{x}; \quad \frac{\partial C}{\partial x} = y - \frac{4V}{x^2}, \quad \frac{\partial C}{\partial y} = x - \frac{5V}{y^2}.$$

We set the two partial derivatives equal to zero and solving the resulting equations simultaneously, by eliminating y ; we get $x^3 = \frac{16V}{5} = 8$, (using $V = 5/2$), so $x = 2$, $y = \frac{5}{2}$, $z = \frac{1}{2}$.

2G. Least-squares Interpolation

2G-1 Find $y = mx + b$ that best fits $(1, 1)$, $(2, 3)$, $(3, 2)$.

$$\begin{aligned} D &= (m + b - 1)^2 + (2m + b - 3)^2 + (3m + b - 2)^2 \\ \frac{\partial D}{\partial m} &= 2(m + b - 1) + 4(2m + b - 3) + 6(3m + b - 2) = 2(14m + 6b - 13) \\ \frac{\partial D}{\partial b} &= 2(m + b - 1) + 2(2m + b - 3) + 2(3m + b - 2) = 2(6m + 3b - 6). \end{aligned}$$

Thus the equations $\frac{\partial D}{\partial m} = 0$ and $\frac{\partial D}{\partial b} = 0$ are $\begin{cases} 14m + 6b = 13 \\ 6m + 3b = 6 \end{cases}$, whose solution is $m = \frac{1}{2}$, $b = 1$, and the line is $y = \frac{1}{2}x + 1$.

2G-4 $D = \sum_i (a + bx_i + cy_i - z_i)^2$. The equations are

$$\begin{aligned} \frac{\partial D}{\partial a} &= \sum 2(a + bx_i + cy_i - z_i) = 0 \\ \frac{\partial D}{\partial b} &= \sum 2x_i(a + bx_i + cy_i - z_i) = 0 \\ \frac{\partial D}{\partial c} &= \sum 2y_i(a + bx_i + cy_i - z_i) = 0 \end{aligned}$$

Cancel the 2's; the equations become (on the right, $\mathbf{x} = [x_1, \dots, x_n]$, $\mathbf{1} = [1, \dots, 1]$, etc.)

$$\begin{aligned} na + (\sum x_i)b + (\sum y_i)c &= \sum z_i & n a + (\mathbf{x} \cdot \mathbf{1}) b + (\mathbf{y} \cdot \mathbf{1}) c &= \mathbf{z} \cdot \mathbf{1} \\ (\sum x_i)a + (\sum x_i^2)b + (\sum x_i y_i)c &= \sum x_i z_i & \text{or } (\mathbf{x} \cdot \mathbf{1}) a + (\mathbf{x} \cdot \mathbf{x}) b + (\mathbf{x} \cdot \mathbf{y}) c &= \mathbf{x} \cdot \mathbf{z} \\ (\sum y_i)a + (\sum x_i y_i)b + (\sum y_i^2)c &= \sum y_i z_i & (\mathbf{y} \cdot \mathbf{1}) a + (\mathbf{x} \cdot \mathbf{y}) b + (\mathbf{y} \cdot \mathbf{y}) c &= \mathbf{y} \cdot \mathbf{z} \end{aligned}$$

2H. Max-min: 2nd Derivative Criterion; Boundary Curves**2H-1**

a) $f_x = 0 : 2x - y = 3; \quad f_y = 0 : -x - 4y = 3$ critical point: $(1, -1)$
 $A = f_{xx} = 2; B = f_{xy} = -1; C = f_{yy} = -4; \quad AC - B^2 = -9 < 0$; saddle point

b) $f_x = 0 : 6x + y = 1; \quad f_y = 0 : x + 2y = 2$ critical point: $(0, 1)$
 $A = f_{xx} = 6; B = f_{xy} = 1; C = f_{yy} = 2; \quad AC - B^2 = 11 > 0$; local minimum

c) $f_x = 0 : 8x^3 - y = 0; \quad f_y = 0 : 2y - x = 0$; eliminating y , we get
 $16x^3 - x = 0$, or $x(16x^2 - 1) = 0 \Rightarrow x = 0, x = \frac{1}{4}, x = -\frac{1}{4}$, giving the critical points
 $(0, 0), (\frac{1}{4}, \frac{1}{8}), (-\frac{1}{4}, -\frac{1}{8})$.

Since $f_{xx} = 24x^2, \quad f_{xy} = -1, \quad f_{yy} = 2$, we get for the three points respectively:

$(0, 0) : \Delta = -1$ (saddle); $(\frac{1}{4}, \frac{1}{8}) : \Delta = 2$ (minimum); $(-\frac{1}{4}, -\frac{1}{8}) : \Delta = 2$ (minimum)

d) $f_x = 0 : 3x^2 - 3y = 0; \quad f_y = 0 : -3x + 3y^2 = 0$. Eliminating y gives
 $-x + x^4 = 0$, or $x(x^3 - 1) = 0 \Rightarrow x = 0, y = 0$ or $x = 1, y = 1$.

Since $f_{xx} = 6x, \quad f_{xy} = -3, \quad f_{yy} = 6y$, we get for the two critical points respectively:

$(0, 0) : AC - B^2 = -9$ (saddle); $(1, 1) : AC - B^2 = 27$ (minimum)

e) $f_x = 0 : 3x^2(y^3 + 1) = 0; \quad f_y = 0 : 3y^2(x^3 + 1) = 0$; solving simultaneously, we get from the first equation that either $x = 0$ or $y = -1$; finding in each case the other coordinate then leads to the two critical points $(0, 0)$ and $(-1, -1)$.

Since $f_{xx} = 6x(y^3 + 1), \quad f_{xy} = 3x^2 \cdot 3y^2, \quad f_{yy} = 6y(x^3 + 1)$, we have

$(-1, -1) : AC - B^2 = -9$ (saddle); $(0, 0) : AC - B^2 = 0$, test fails.

(By studying the behavior of $f(x, y)$ on the lines $y = mx$, for different values of m , it is possible to see that also $(0, 0)$ is a saddle point.)

2H-3 The region R has no critical points; namely, the equations $f_x = 0$ and $f_y = 0$ are

$$2x + 2 = 0, \quad 2y + 4 = 0 \Rightarrow x = -1, y = -2,$$

but this point is not in R . We therefore investigate the diagonal boundary of R , using the parametrization $x = t, y = -t$. Restricted to this line, $f(x, y)$ becomes a function of t alone, which we denote by $g(t)$, and we look for its maxima and minima.

$$g(t) = f(t, -t) = 2t^2 - 4t - 1; \quad g'(t) = 4t - 2, \text{ which is } 0 \text{ at } t = 1/2.$$

This point is evidently a minimum for $g(t)$; there is no maximum: $g(t)$ tends to ∞ . Therefore for $f(x, y)$ on R , the minimum occurs at the point $(1/2, -1/2)$, and there is no maximum; $f(x, y)$ tends to infinity in different directions in R .

2H-4 We have $f_x = y - 1$, $f_y = x - 1$, so the only critical point is at $(1, 1)$.

a) On the two sides of the boundary, the function $f(x, y)$ becomes respectively

$$y = 0: f(x, y) = -x + 2; \quad x = 0: f(x, y) = -y + 2.$$

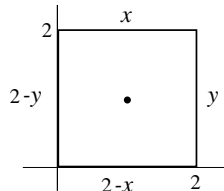
Since the function is linear and decreasing on both sides, it has no minimum points (informally, the minimum is $-\infty$). Since $f(1, 1) = 1$ and $f(x, x) = x^2 - 2x + 2 \rightarrow \infty$ as $x \rightarrow \infty$, the maximum of f on the first quadrant is ∞ .

b) Continuing the reasoning of (a) to find the maximum and minimum points of $f(x, y)$ on the boundary, on the other two sides of the boundary square, the function $f(x, y)$ becomes

$$y = 2: f(x, y) = x \quad x = 2: f(x, y) = y$$

Since $f(x, y)$ is thus increasing or decreasing on each of the four sides, the maximum and minimum points on the boundary square R can only occur at the four corner points; evaluating $f(x, y)$ at these four points, we find

$$f(0, 0) = 2; \quad f(2, 2) = 2; \quad f(2, 0) = 0; \quad f(0, 2) = 0.$$



As in (a), since $f(1, 1) = 1$,

maximum points of f on R : $(0, 0)$ and $(2, 2)$; minimum points: $(2, 0)$ and $(0, 2)$.

c) The data indicates that $(1, 1)$ is probably a saddle point. Confirming this, we have $f_{xx} = 0$, $f_{xy} = 1$, $f_{yy} = 0$ for all x and y ; therefore $AC - B^2 = -1 < 0$, so $(1, 1)$ is a saddle point, by the 2nd-derivative criterion.

2H-5 Since $f(x, y)$ is linear, it will not have critical points: namely, for all x and y we have $f_x = 1$, $f_y = \sqrt{3}$. So any maxima or minima must occur on the boundary circle.

We parametrize the circle by $x = \cos \theta$, $y = \sin \theta$; restricted to this boundary circle, $f(x, y)$ becomes a function of θ alone which we call $g(\theta)$:

$$g(\theta) = f(\cos \theta, \sin \theta) = \cos \theta + \sqrt{3} \sin \theta + 2.$$

Proceeding in the usual way to find the maxima and minima of $g(\theta)$, we get

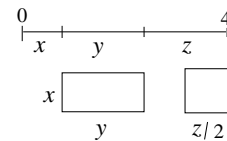
$$g'(\theta) = -\sin \theta + \sqrt{3} \cos \theta = 0, \quad \text{or} \quad \tan \theta = \sqrt{3}.$$

It follows that the two critical points of $g(\theta)$ are $\theta = \frac{\pi}{3}$ and $\frac{4\pi}{3}$; evaluating g at these two points, we get $g(\pi/3) = 4$ (the maximum), and $g(4\pi/3) = 0$ (the minimum).

Thus the maximum of $f(x, y)$ in the circular disc R is at $(1/2, \sqrt{3}/2)$, while the minimum is at $(-1/2, -\sqrt{3}/2)$.

2H-6 a) Since $z = 4 - x - y$, the problem is to find on R the maximum and minimum of the total area

$$f(x, y) = xy + \frac{1}{4}(4 - x - y)^2$$



where R is the triangle given by $R: 0 \leq x, 0 \leq y, x + y \leq 4$.

To find the critical points of $f(x, y)$, the equations $f_x = 0$ and $f_y = 0$ are respectively

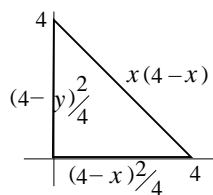
$$y - \frac{1}{2}(4 - x - y) = 0; \quad x - \frac{1}{2}(4 - x - y) = 0,$$

which imply first that $x = y$, and from this, $x - \frac{1}{2}(4 - 2x)$; the unique solution is $x = 1$, $y = 1$.

The region R is a triangle, on whose sides $f(x, y)$ takes respectively the values

$$\begin{aligned} \text{bottom: } y = 0; \quad f &= \frac{1}{4}(4-x)^2; & \text{left side: } x = 0; \quad f &= \frac{1}{4}(4-y)^2; \\ \text{diagonal } y = 4-x; \quad f &= x(4-x). \end{aligned}$$

On the bottom and side, f is decreasing; on the diagonal, f has a maximum at $x = 2, y = 2$. Therefore we need to examine the three corner points and $(2, 2)$ as candidates for maximum and minimum points, as well as the critical point $(1, 1)$. We find



$$f(0,0) = 4; \quad f(4,0) = 0; \quad f(0,4) = 0; \quad f(2,2) = 4 \quad f(1,1) = 2.$$

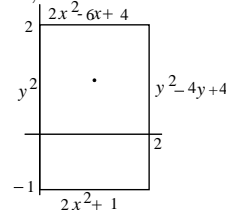
It follows that the critical point is just a saddle point; to get the maximum total area 4, make $x = y = 0, z = 4$, or $x = y = 2, z = 0$, either of which gives a point “rectangle” and a square of side 2; for the minimum total area 0, take for example $x = 0, y = 4, z = 0$, which gives a “rectangle” of length 4 with zero area, and a point square.

b) We have $f_{xx} = \frac{1}{2}, f_{xy} = \frac{3}{2}, f_{yy} = \frac{1}{2}$ for all x and y ; therefore $AC - B^2 = -2 < 0$, so $(1, 1)$ is a saddle point, by the 2nd-derivative criterion.

2H-7 a) $f_x = 4x - 2y - 2, f_y = -2x + 2y$; setting these = 0 and solving simultaneously, we get $x = 1, y = 1$, which is therefore the only critical point.

On the four sides of the boundary rectangle R , the function $f(x, y)$ becomes:

$$\begin{aligned} \text{on } y = -1: \quad f(x, y) &= 2x^2 + 1; & \text{on } y = 2: \quad f(x, y) &= 2x^2 - 6x + 4 \\ \text{on } x = 0: \quad f(x, y) &= y^2; & \text{on } x = 2: \quad f(x, y) &= y^2 - 4y + 4 \end{aligned}$$



By one-variable calculus, $f(x, y)$ is increasing on the bottom and decreasing on the right side; on the left side it has a minimum at $(0, 0)$, and on the top a minimum at $(\frac{3}{2}, 2)$. Thus the maximum and minimum points on the boundary rectangle R can only occur at the four corner points, or at $(0, 0)$ or $(\frac{3}{2}, 2)$. At these we find:

$$f(0, -1) = 1; \quad f(0, 2) = 4; \quad f(2, -1) = 9; \quad f(2, 2) = 0; \quad f(\frac{3}{2}, 2) = -\frac{1}{2}, \quad f(0, 0) = 0.$$

At the critical point $f(1, 1) = -1$; comparing with the above, it is a minimum; therefore, maximum point of $f(x, y)$ on R : $(2, -1)$ minimum point of $f(x, y)$ on R : $(1, 1)$

b) We have $f_{xx} = 4, f_{xy} = -2, f_{yy} = 2$ for all x and y ; therefore $AC - B^2 = 4 > 0$ and $A = 4 > 0$, so $(1, 1)$ is a minimum point, by the 2nd-derivative criterion.

2I. Lagrange Multipliers

2I-1 Letting $P : (x, y, z)$ be the point, in both problems we want to maximize $V = xyz$, subject to a constraint $f(x, y, z) = c$. The Lagrange equations for this, in vector form, are

$$\nabla(xyz) = \lambda \cdot \nabla f(x, y, z), \quad f(x, y, z) = c.$$

a) Here $f = c$ is $x + 2y + 3z = 18$; equating components, the Lagrange equations become

$$yz = \lambda, \quad xz = 2\lambda, \quad xy = 3\lambda; \quad x + 2y + 3z = 18.$$

To solve these symmetrically, multiply the left sides respectively by x, y , and z to make them equal; this gives

$$\lambda x = 2\lambda y = 3\lambda z, \quad \text{or} \quad x = 2y = 3z = 6, \quad \text{since the sum is 18.}$$

We get therefore as the answer $x = 6$, $y = 3$, $z = 2$. This is a maximum point, since if P lies on the triangular boundary of the region in the first octant over which it varies, the volume of the box is zero.

b) Here $f = c$ is $x^2 + 2y^2 + 4z^2 = 12$; equating components, the Lagrange equations become

$$yz = \lambda \cdot 2x, \quad xz = \lambda \cdot 4y, \quad xy = \lambda \cdot 8z; \quad x^2 + 2y^2 + 4z^2 = 12.$$

To solve these symmetrically, multiply the left sides respectively by x , y , and z to make them equal; this gives

$$\lambda \cdot 2x^2 = \lambda \cdot 4y^2 = \lambda \cdot 8z^2, \quad \text{or} \quad x^2 = 2y^2 = 4z^2 = 4, \quad \text{since the sum is 12.}$$

We get therefore as the answer $x = 2$, $y = \sqrt{2}$, $z = 1$. This is a maximum point, since if P lies on the boundary of the region in the first octant over which it varies (1/8 of the ellipsoid), the volume of the box is zero.

2I-2 Since we want to minimize $x^2 + y^2 + z^2$, subject to the constraint $x^3y^2z = 6\sqrt{3}$, the Lagrange multiplier equations are

$$2x = \lambda \cdot 3x^2y^2z, \quad 2y = \lambda \cdot 2x^3yz, \quad 2z = \lambda \cdot x^3y^2; \quad x^3y^2z = 6\sqrt{3}.$$

To solve them symmetrically, multiply the first three equations respectively by x , y , and z , then divide them through respectively by 3, 2, and 1; this makes the right sides equal, so that, after canceling 2 from every numerator, we get

$$\frac{x^2}{3} = \frac{y^2}{2} = z^2; \quad \text{therefore} \quad x = z\sqrt{3}, \quad y = z\sqrt{2}.$$

Substituting into $x^3y^2z = 6\sqrt{3}$, we get $3\sqrt{3}z^3 \cdot 2z^2 \cdot z = 6\sqrt{3}$, which gives as the answer, $x = \sqrt{3}$, $y = \sqrt{2}$, $z = 1$.

This is clearly a minimum, since if P is near one of the coordinate planes, one of the variables is close to zero and therefore one of the others must be large, since $x^3y^2z = 6\sqrt{3}$; thus P will be far from the origin.

2I-3 Referring to the solution of 2F-2, we let x be the length of the ends, y the length of the sides, and z the height, and get

$$\text{total area of cardboard } A = 3xy + 4xz + 2yz, \quad \text{volume } V = xyz = 1.$$

The Lagrange multiplier equations $\nabla A = \lambda \cdot \nabla(xyz)$; $xyz = 1$, then become

$$3y + 4z = \lambda yz, \quad 3x + 2z = \lambda xz, \quad 4x + 2y = \lambda xy, \quad xyz = 1.$$

To solve these equations for x, y, z, λ , treat them symmetrically. Divide the first equation through by yz , and treat the next two equations analogously, to get

$$3/z + 4/y = \lambda, \quad 3/z + 2/x = \lambda, \quad 4/y + 2/x = \lambda,$$

which by subtracting the equations in pairs leads to $3/z = 4/y = 2/x$; setting these all equal to k , we get $x = 2/k, y = 4/k, z = 3/k$, which shows the proportions using least cardboard are $x : y : z = 2 : 4 : 3$.

To find the actual values of x, y , and z , we set $1/k = m$; then substituting into $xyz = 1$ gives $(2m)(4m)(3m) = 1$, from which $m^3 = 1/24$, $m = 1/2 \cdot 3^{1/3}$, giving finally

$$x = \frac{1}{3^{1/3}}, \quad y = \frac{2}{3^{1/3}}, \quad z = \frac{3}{2 \cdot 3^{1/3}}.$$

2I-4 The equations for the cost C and the volume V are $xy + 4yz + 6xz = C$ and $xyz = V$. The Lagrange multiplier equations for the two problems are

$$\text{a)} \quad yz = \lambda(y + 6z), \quad xz = \lambda(x + 4z), \quad xy = \lambda(4y + 6x); \quad xy + 4yz + 6xz = 72$$

$$\text{b)} \quad y + 6z = \mu \cdot yz, \quad x + 4z = \mu \cdot xz, \quad 4y + 6x = \mu \cdot xy; \quad xyz = 24$$

The first three equations are the same in both cases, since we can set $\mu = 1/\lambda$. Solving the first three equations in (a) symmetrically, we multiply the equations through by x , y , and z respectively, which makes the left sides equal; since the right sides are therefore equal, we get after canceling the λ ,

$$xy + 6xz = xy + 4yz = 4yz + 6xz, \quad \text{which implies} \quad xy = 4yz = 6xz.$$

a) Since the sum of the three equal products is 72, by hypothesis, we get

$$xy = 24, \quad yz = 6, \quad xz = 4;$$

from the first two we get $x = 4z$, and from the first and third we get $y = 6z$, which lead to the solution $x = 4$, $y = 6$, $z = 1$.

b) Dividing $xy = 4yz = 6xz$ by xyz leads after cross-multiplication to $x = 4z$, $y = 6z$; since by hypothesis, $xyz = 24$, again this leads to the solution $x = 4$, $y = 6$, $z = 1$.

2J. Non-independent Variables

2J-1 a) $\left(\frac{\partial w}{\partial y}\right)_z$ means that x is the dependent variable; get rid of it by writing

$$w = z - y^2 + y^2 + z^2 = z + z^2. \quad \text{This shows that} \quad \left(\frac{\partial w}{\partial y}\right)_z = 0.$$

b) To calculate $\left(\frac{\partial w}{\partial z}\right)_y$, once again x is the dependent variable; as in part (a), we have $w = z + z^2$ and so $\left(\frac{\partial w}{\partial z}\right)_y = 1 + 2z$.

2J-2 a) Differentiating $z = x^2 + y^2$ w.r.t. y : $0 = 2x \left(\frac{\partial x}{\partial y}\right)_z + 2y$; so $\left(\frac{\partial x}{\partial y}\right)_z = -\frac{y}{x}$;

By the chain rule, $\left(\frac{\partial w}{\partial y}\right)_z = 2x \left(\frac{\partial x}{\partial y}\right)_z + 2y = 2x \left(\frac{-y}{x}\right) + 2y = 0$.

Differentiating $z = x^2 + y^2$ with respect to z : $1 = 2x \left(\frac{\partial x}{\partial z}\right)_y$; so $\left(\frac{\partial x}{\partial z}\right)_y = \frac{1}{2x}$;

By the chain rule, $\left(\frac{\partial w}{\partial z}\right)_y = 2x \left(\frac{\partial x}{\partial z}\right)_y + 2z = 1 + 2z$.

b) Using differentials, $dw = 2xdx + 2ydy + 2zdz$, $dz = 2xdx + 2ydy$; since the independent variables are y and z , we eliminate dx by subtracting the second equation from the first, which gives

$$dw = 0 dy + (1 + 2z) dz;$$

therefore by **D2**, we get $\left(\frac{\partial w}{\partial y}\right)_z = 0$, $\left(\frac{\partial w}{\partial z}\right)_y = 1 + 2z$.

2J-3 a) To calculate $\left(\frac{\partial w}{\partial t}\right)_{x,z}$, we see that y is the dependent variable; solving for it, we get $y = \frac{zt}{x}$; using the chain rule, $\left(\frac{\partial w}{\partial t}\right)_{x,z} = x^3 \left(\frac{\partial y}{\partial t}\right)_{x,z} - z^2 = x^3 \frac{z}{x} - z^2 = x^2 z - z^2$.

b) Similarly, $\left(\frac{\partial w}{\partial z}\right)_{x,y}$ means that t is the dependent variable; since $t = \frac{xy}{z}$, we have by the chain rule, $\left(\frac{\partial w}{\partial z}\right)_{x,y} = -2zt - z^2 \left(\frac{\partial t}{\partial z}\right)_{x,y} = -2zt - z^2 \cdot \frac{-xy}{z^2} = -zt$.

2J-4 The differentials are calculated in equation (4).

a) Since x, z, t are independent, we eliminate dy by solving the second equation for $x dy$, substituting this into the first equation, and grouping terms:

$$dw = 2x^2 y dx + (x^2 z - z^2) dt + (x^2 t - 2zt) dz, \text{ which shows by } \mathbf{D2} \text{ that } \left(\frac{\partial w}{\partial t}\right)_{x,z} = x^2 z - z^2.$$

b) Since x, y, z are independent, we eliminate dt by solving the second equation for $z dt$, substituting this into the first equation, and grouping terms:

$$dw = (3x^2 y - zy) dx + (x^3 - zx) dy - zt dz, \text{ which shows by } \mathbf{D2} \text{ that } \left(\frac{\partial w}{\partial z}\right)_{x,y} = -zt.$$

2J-5 a) If $pv = nRT$, then $\left(\frac{\partial S}{\partial p}\right)_v = S_p + S_T \cdot \left(\frac{\partial T}{\partial p}\right)_v = S_p + S_T \cdot \frac{v}{nR}$.

b) Similarly, we have $\left(\frac{\partial S}{\partial T}\right)_v = S_T + S_p \cdot \left(\frac{\partial p}{\partial T}\right)_v = S_T + S_p \cdot \frac{nR}{v}$.

2J-6 a) $\left(\frac{\partial w}{\partial u}\right)_x = 3u^2 - v^2 - u \cdot 2v \left(\frac{\partial v}{\partial u}\right)_x = 3u^2 - v^2 - 2uv$.

$$\left(\frac{\partial w}{\partial x}\right)_u = -u \cdot 2v \left(\frac{\partial v}{\partial x}\right)_u = -2uv.$$

b) $dw = (3u^2 - v^2)du - 2uv dv$; $du = x dy + y dx$; $dv = du + dx$;
for both derivatives, u and x are the independent variables, so we eliminate dv , getting

$$dw = (3u^2 - v^2)du - 2uv(du + dx) = (3u^2 - v^2 - 2uv)du - 2uv dx,$$

whose coefficients by **D2** are $\left(\frac{\partial w}{\partial u}\right)_x$ and $\left(\frac{\partial w}{\partial x}\right)_u$.

2J-7 Since we need both derivatives for the gradient, we use differentials.

$$df = 2dx + dy - 3dz \quad \text{at } P; \quad dz = 2x dx + dy = 2dx + dy \quad \text{at } P;$$

the independent variables are to be x and z , so we eliminate dy , getting

$$df = 0 dx - 2 dz \quad \text{at the point } (x, z) = (1, 1). \quad \text{So } \nabla g = \langle 0, -2 \rangle \text{ at } (1, 1).$$

2J-8 To calculate $\left(\frac{\partial w}{\partial r}\right)_\theta$, note that r and θ are independent. Therefore,

$$\left(\frac{\partial w}{\partial r}\right)_\theta = \frac{\partial w}{\partial r} + \frac{\partial w}{\partial x} \cdot \left(\frac{\partial x}{\partial r}\right)_\theta. \quad \text{Now, } x = r \cos \theta, \text{ so } \left(\frac{\partial x}{\partial r}\right)_\theta = \cos \theta. \quad \text{Therefore}$$

$$\left(\frac{\partial w}{\partial r}\right)_\theta = \frac{r}{\sqrt{r^2 - x^2}} + \frac{-x}{\sqrt{r^2 - x^2}} \cdot \cos \theta = \frac{r - x \cos \theta}{\sqrt{r^2 - x^2}}$$

$$= \frac{r - r \cos^2 \theta}{r |\sin \theta|} = \frac{r \sin^2 \theta}{r |\sin \theta|} = |\sin \theta|.$$

2K. Partial Differential Equations

2K-1 $w = \frac{1}{2} \ln(x^2 + y^2)$. If $(x, y) \neq (0, 0)$, then

$$\begin{aligned} w_{xx} &= \frac{\partial}{\partial x}(w_x) = \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \\ w_{yy} &= \frac{\partial}{\partial y}(w_y) = \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) = \frac{x^2 - y^2}{(x^2 + y^2)^2}, \end{aligned}$$

Therefore w satisfies the two-dimensional Laplace equation, $w_{xx} + w_{yy} = 0$; we exclude the point $(0, 0)$ since $\ln 0$ is not defined.

2K-2 If $w = (x^2 + y^2 + z^2)^n$, then $\frac{\partial}{\partial x}(w_x) = \frac{\partial}{\partial x}(2x \cdot n(x^2 + y^2 + z^2)^{n-1})$

$$= 2n(x^2 + y^2 + z^2)^{n-1} + 4x^2 n(n-1)(x^2 + y^2 + z^2)^{n-2}$$

We get w_{yy} and w_{zz} by symmetry; adding and combining, we get

$$\begin{aligned} w_{xx} + w_{yy} + w_{zz} &= 6n(x^2 + y^2 + z^2)^{n-1} + 4(x^2 + y^2 + z^2)n(n-1)(x^2 + y^2 + z^2)^{n-2} \\ &= 2n(2n+1)(x^2 + y^2 + z^2)^{n-1}, \text{ which is identically zero if } n = 0, \text{ or if } n = -1/2. \end{aligned}$$

2K-3 a) $w = ax^2 + bxy + cy^2$; $w_{xx} = 2a$, $w_{yy} = 2c$.

$$w_{xx} + w_{yy} = 0 \Rightarrow 2a + 2c = 0, \text{ or } c = -a.$$

Therefore all quadratic polynomials satisfying the Laplace equation are of the form

$$ax^2 + bxy - ay^2 = a(x^2 - y^2) + bxy;$$

i.e., linear combinations of the two polynomials $f(x, y) = x^2 - y^2$ and $g(x, y) = xy$.

2K-4 The one-dimensional wave equation is $w_{xx} = \frac{1}{c^2} w_{tt}$. So

$$\begin{aligned} w &= f(x + ct) + g(x - ct) \Rightarrow w_{xx} = f''(x + ct) + g''(x - ct) \\ &\Rightarrow w_t = cf'(x + ct) - cg'(x - ct). \\ &\Rightarrow w_{tt} = c^2 f''(x + ct) + c^2 g''(x - ct) = c^2 w_{xx}, \end{aligned}$$

which shows w satisfies the wave equation.

2K-5 The one-dimensional heat equation is $w_{xx} = \frac{1}{\alpha^2} w_t$. So if $w(x, t) = \sin kx e^{rt}$, then

$$\begin{aligned} w_{xx} &= e^{rt} \cdot k^2 (-\sin kx) = -k^2 w. \\ w_t &= r e^{rt} \sin kx = r w. \end{aligned}$$

Therefore, we must have $-k^2 w = \frac{1}{\alpha^2} r w$, or $r = -\alpha^2 k^2$.

However, from the additional condition that $w = 0$ at $x = 1$, we must have

$$\sin k e^{rt} = 0;$$

Therefore $\sin k = 0$, and so $k = n\pi$, where n is an integer.

To see what happens to w as $t \rightarrow \infty$, we note that since $|\sin kx| \leq 1$,

$$|w| = e^{rt} |\sin kx| \leq e^{rt}.$$

Now, if $k \neq 0$, then $r = -\alpha^2 k^2$ is negative and $e^{rt} \rightarrow 0$ as $t \rightarrow \infty$; therefore $|w| \rightarrow 0$.

Thus w will be a solution satisfying the given side conditions if $k = n\pi$, where n is a non-zero integer, and $r = -\alpha^2 k^2$.