Concourse 18.02 Problem Set 9 – Fall 2018

due Tues, Nov 20

Read sections:

14.8 (Surface Area)
15.1 (Vector Fields, Divergence and Curl)
Notes V1 (Plane Vector Fields)
Notes V8 (Vector Fields in Space)

15.2 (Line Integrals)
Notes V11 (Line Integrals in Space)

15.3 (Fundamental Theorem and Independence of Path)
Notes V2 (Gradient Fields and Exact Differentials)
Notes V12 (Gradient Fields in Space)

15.4 (Green's Theorem)

Do the following problems:

1. [14.8/12] Show by integration that the surface area of the (inverted) conical surface $z = br$ between the planes $z = 0$ (its vertex) and $z = h = ab$ (its base) is given by $A = \pi aL$, where L is the slant height $\sqrt{a^2 + h^2}$ and a is the radius of the base of the cone.

2. [14.8/16] Find the area of the part of the sphere $r^2 + z^2 = a^2$ that lies within the cylinder $r = a\sin\theta$.

3. [15.1/21] Calculate the divergence and curl of the vector field $\mathbf{F}(x, y, z) = (y^2 + z^2)\,\mathbf{i} + (x^2 + z^2)\,\mathbf{j} + (x^2 + y^2)\,\mathbf{k}$.

4. [15.2/9] Evaluate $\int_C x^2 y\,dx + xy^3 dy$ where C consists of the line segments from $(-1,1)$ to $(2,1)$ and from $(2,1)$ to $(2,5)$.

5. [15.2/13] Evaluate $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ where $\mathbf{F}(x, y, z) = y\,\mathbf{i} - x\,\mathbf{j} + z\,\mathbf{k}$ and C is given parametrically by $x = \sin t$, $y = \cos t$, $z = 2t$, $0 \le t \le \pi$.

6. [15.2/16] Evaluate $\int_C xyz\,ds$ where C is the straight line segment from $(1,-1,2)$ to $(3,2,5)$.

7. [15.2/33a] Imagine an infinitely long and uniformly charged wire that coincides with the z-axis. The electric force that it exerts on a unit charge at the point $(x, y) \ne (0,0)$ in the xy-plane is $\mathbf{F}(x, y) = \dfrac{k(x\,\mathbf{i} + y\,\mathbf{j})}{x^2 + y^2}$. [Can you show why??] Find the work done by $\mathbf{F}$ in moving a unit charge along the straight line segment from $(1,0)$ to $(1,1)$.

8. [SN-4B/2] For the following fields $\mathbf{F}$ and curves C, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ without any formal calculation, appealing instead to the geometry of $\mathbf{F}$ and C.
   (a) $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$; C is the counterclockwise circle, center at $(0,0)$, radius a.
   (b) $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j}$; C is the counterclockwise circle, center at $(0,0)$, radius a.

9. [15.3/3] Determine whether the vector field $\mathbf{F}(x, y) = (3x^2 + 2y^2)\,\mathbf{i} + (4xy + 6y^2)\,\mathbf{j}$ is conservative. If so, find a potential function (either by inspection – *guess and check* – or by using a more formal approach).

10. [15.3/34] Let $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + (xz + y)\,\mathbf{j} + (xy + 1)\,\mathbf{k}$. Define the function f by $f(x, y, z) = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$ where C is the straight line segment from $(0,0,0)$ to (x, y, z). Determine f by evaluating this line integral, then show that $\overrightarrow{\nabla f} = \mathbf{F}$.

11. [15.3/36] Let $\mathbf{F} = k\mathbf{r}/r^3$ be the inverse-square force field of Example 7 in Section 15.2. $\mathbf{F}(x,y,z) = \frac{k\mathbf{r}}{r^3} = \frac{k(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})}{(x^2+y^2+z^2)^{3/2}}$. Show that the work done by F in moving a particle from a point at a distance r_1 from the origin to a point at a distance r_2 from the origin is given by $W = k\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$.

12. [SN-4C/5] For each of the following, tell for what value of the constants the field will be a gradient field, and for this value, find the corresponding (mathematical) potential function.

    a) $\mathbf{F} = (y^2 + 2x)\,\mathbf{i} + axy\,\mathbf{j}$ b) $\mathbf{F} = e^{x+y}((x+a)\,\mathbf{i} + x\,\mathbf{j})$

13. [15.4/4] Apply Green's Theorem to evaluate the integral $\oint_C (x^2 - y^2)\,dx + xy\,dy$ around the positively oriented curve C that is the boundary of the region bounded by the line $y = x$ and the parabola $y = x^2$.

14. [15.4/6] Apply Green's Theorem to evaluate the integral $\oint_C y^2\,dx + (2x - 3y)\,dy$ counterclockwise around the circle $x^2 + y^2 = 9$.

15. [15.4/9] Apply Green's Theorem to evaluate the integral $\oint_C y^2\,dx + xy\,dy$ where C is the positively oriented ellipse with equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

16. [15.4/29] Let R be the plane region with area A enclosed by the positively oriented piecewise-smooth simple closed curve C (in the xy-plane). Use Green's Theorem to show that the coordinates of the centroid $(\overline{x}, \overline{y})$ of R are

$$\overline{x} = \frac{1}{2A}\oint_C x^2 dy \qquad \overline{y} = -\frac{1}{2A}\oint_C y^2 dx$$

17. [SN-4D/3] Find the area inside the hypocycloid $x^{2/3} + y^{2/3} = 1$, by using Green's theorem. (This curve can be parameterized by $x = \cos^3\theta$, $y = \sin^3\theta$, between suitable limits on θ.)