# Concourse 18.02 Problem Set 7 – Fall 2018

due Thursday, November 1

Read Supplementary Notes N (Non-independent Variables) and text sections 14.1 (Double Integrals), 14.2 (Double Integrals Over More General Regions), 14.3 (Area and Volume by Double Integration), 14.4 (Double Integrals in Polar Coordinates), and 14.5 (Applications of Double Integrals).

**Non-independent Variables**

1. [SN-2J/3] In Example 2 [with $w = x^3y - z^2t$ and $xy = zt$], using the chain rule calculate, in terms of *x*, *y*, *z*, *t*, the derivatives

   (a) $\left(\dfrac{\partial w}{\partial t}\right)_{x,z}$ (b) $\left(\dfrac{\partial w}{\partial z}\right)_{x,y}$

2. [SN-2J/4] Repeat 2J-3 (Problem 1 above), doing the calculation using differentials.

3. [SN-2J/5] Let $S = S(p,v,T)$ be the entropy of a gas, assumed to obey the ideal gas law [ $pv = nRT$ where *p* represents the pressure of the gas, *v* represents the volume of the gas, *T* represents the absolute temperature of the gas, and *R* is a universal constant]. Give expressions in terms of the formal partial derivatives $S_p$, $S_v$, and $S_T$ for the derivatives:

   (a) $\left(\dfrac{\partial S}{\partial p}\right)_{v}$ (b) $\left(\dfrac{\partial S}{\partial T}\right)_{v}$

**Double Integrals**

4. [14.2/22] Evaluate the integral of the function $f(x,y) = xy$ over the plane region *R* that is the first-quadrant quarter-circle bounded by $x^2 + y^2 = 1$ and the coordinate axes.

5. [14.2/24] Evaluate the integral of the function $f(x,y) = 9 - y$ over the plane region *R* that is the triangle with vertices $(0,0)$, $(0,9)$, and $(3,6)$.

6. [14.2/30] Given the integral $\int_0^1\int_y^1 e^{-x^2}\,dx\,dy$, first sketch the region of integration, then reverse the order of integration and evaluate the resulting integral.

7. [14.2/33] Given the integral $\int_0^1\int_y^1 \left(\dfrac{1}{1+x^4}\right)dx\,dy$, first sketch the region of integration, then reverse the order of integration and evaluate the resulting integral.

**Area and Volume by Double Integrals**

8. [14.3/4] Use double integration to find the area of the region in the *xy*-plane bounded by the curves $y = 2x + 3$ and $y = 6x - x^2$.

9. [14.3/16] Find the volume of the solid that lies below the surface $z = 3x + 2y$ and above the region in the *xy*-plane bounded by the curves $x = 0$, $y = 0$, and $x + 2y = 4$.

10. [14.3/20] Find the volume of the solid that lies below the surface $z = y^2$ and above the region in the *xy*-plane bounded by the curves $x = y^2$ and $x = 4$.

11. [14.3/37] Find the volume of the first octant part of the solid bounded by the cylinders $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$. [*Suggestion*: One order of integration is considerably easier than the other.]

**Double Integrals in Polar Coordinates**

12 [14.4/14] Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{4-x^2-y^2}}\, dy\, dx$ by first converting to polar coordinates.

[Draw the region first!]

13. [14.4/29] Find the volume of the "ice-cream cone" bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the cone $z = \sqrt{x^2 + y^2}$ .

14. [14.4/39] Use double integrals in polar coordinates to find the volume of the solid bounded by the elliptical paraboloids $z = x^2 + 2y^2$ and $z = 12 - 2x^2 - y^2$ .

**Applications of Double Integrals**

15. [14.5/24] Find the mass and centroid (center of mass) of the plane lamina (thin sheet) bounded by $y = x^2$ and $y = 2x + 3$ with density given by $\delta(x, y) = x^2$ .

16. [14.5/57] Find the mass and centroid $(\overline{x}, \overline{y})$ of the plane lamina bounded by the circle with polar equation $r = 2\sin\theta$ with density given by $\delta(x, y) = y$ .

**Miscellaneous**

17. The equation of a surface of revolution obtained by spinning the (1-dimensional) graph of the function $z = f(y)$ for $0 \le a \le y \le b$ in the *yz*-plane around the *z*-axis is given in polar coordinates by $z = f(r, \theta) = f(r)$ (that is, there is no dependence on $\theta$). Assume that $f(y) > 0$ for $a \le y \le b$.

a) Show that the formula obtained by using the double integral in polar coordinates for the volume under the graph of this surface of revolution and over the *xy*-plane is the same as that given by the "shell method" in single-variable calculus.

b) Illustrate with a sketch.

18. a) We define the improper integral (over the entire plane $\mathbf{R}^2$)

$$I = \iint_{\mathbf{R}^2} e^{-(x^2+y^2)} dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dy\, dx = \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)} dA$$

where $D_a$ is the disk with radius *a* and center the origin. Show that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dy\, dx = \pi$ .

b) An equivalent definition of the improper integral in part (a) is $I = \iint_{\mathbf{R}^2} e^{-(x^2+y^2)} dA = \lim_{a\to\infty} \iint_{S_a} e^{-(x^2+y^2)} dA$ where $S_a$ is the square with vertices $(\pm a, \pm a)$. Use this to show that $\int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \pi$ .

c) Deduce that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ .

d) By making an appropriate change of variable, show that $\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$ . [This is a fundamental result for probability and statistics, specifically the normal probability distribution.]