Concourse 18.02 Problem Set 6 – Fall 2018

due no later than <u>Thurs, October 18</u>

Read sections 13.5 (Multivariable Optimization Problems), 13.9 (Lagrange Multipliers and Constrained Optimization), 13.10 (Critical Points of Functions of Two Variables and the Second Derivative Test); and Supplementary Notes LS (Least Squares Interpolation) and do the following problems. <u>You may use any method</u> unless a particular method is specified.

1. [13.5/24] Find the maximum and minimum values of the function $f(x, y) = x^2 + y^2 - x$ on the plane region *R* that is the square with vertices at $(\pm 1, \pm 1)$.

2. [13.5/38] Find the dimensions of the open-topped (rectangular) box with volume 4000 cm^3 whose bottom and four sides have minimal total surface area.

3. [13.5/40] Find the relative dimensions of the closed rectangular box of fixed volume that minimizes the total cost of the material to construct the box if the material for its top and bottom costs $3/$ft^2$ and the material for its four sides costs $4/$ft^2$.

4. [13.5/46] Three sides of a rectangular box lie in the coordinate planes, their common vertex at the origin; the opposite vertex on the plane with equation $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ (where *a*, *b*, and *c* are positive constants). In terms of *a*, *b*, and *c*, what is the maximum possible volume of such a box?

5. [13.9/8] Find the maximum and minimum values – if any – of the function $f(x, y, z) = 3x + 2y + z$ subject to the constraint that $x^2 + y^2 + z^2 = 1$.

6. [13.9/22] Use the Method of Lagrange Multipliers to find the first-octant point $P(x, y, z)$ on the surface $2x + 2y + z = 27$ closest to the fixed point $Q(9, 9, 9)$. [*Suggestion*: Minimize the squared distance expressed as a function of *x* and *y*.]

7. [13.9/36] A triangle with sides *x*, *y*, and *z* has fixed perimeter $2s = x + y + z$. Its area is given by Heron's formula: $A = \sqrt{s(s-x)(s-y)(s-z)}$. Use the Method of Lagrange Multipliers to show that, among all triangles with the given perimeter, the one of largest area is equilateral.
[*Suggestion*: Consider maximizing A^2 rather than *A*.]

8. [13.9/41] Find the highest and lowest points on the ellipse formed by the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $2x + y - z = 4$.

9. Suppose that production (measured in lollipops) at a small lollipop factory is modeled by the Cobb-Douglas function $p(x, y) = 500x^{0.7}y^{0.5}$ where *x* is the number of units of labor and *y* is the number of units of capital. Further suppose that the cost of labor is $35 per unit and the cost of capital is $16 per unit.

   a) Find the least costly combination of labor and capital needed to produce 40,000 lollipops and find this minimum cost.

   b) Now find what combination of labor and capital will yield the maximum number of lollipops with a fixed budget of $4800 and find this maximum number of lollipops.

10 a) Maximize $\sum_{i=1}^{n} x_i y_i$ subject to the constraints $\sum_{i=1}^{n} x_i^2 = 1$ and $\sum_{i=1}^{n} y_i^2 = 1$.

b) Put $x_i = \frac{a_i}{\sqrt{\sum a_j^2}}$ and $y_i = \frac{b_i}{\sqrt{\sum b_j^2}}$ to show that $\sum a_i b_i \le \sqrt{\sum a_j^2}\sqrt{\sum b_j^2}$ for any numbers $a_1, \dots a_n, b_1, \dots b_n$. This inequality is known as the **Cauchy-Schwartz Inequality**. [*Note*: In vector notation, if we write $\mathbf{a} = \langle a_1, \dots, a_n \rangle$ and $\mathbf{b} = \langle b_1, \dots, b_n \rangle$ and if we use Lagrange Multipliers to determine both the maximum and minimum values, this says that $|\mathbf{a} \cdot \mathbf{b}| \le \|\mathbf{a}\| \|\mathbf{b}\|$, where $\mathbf{a} \cdot \mathbf{b}$ is the dot product.]

11. Find the maximum and minimum volumes of a rectangular box whose surface area is 1500 cm^2 and whose total edge length is 200 cm. [*Note*: Due to the multiple constraints, it's not hard to show that in neither case will the box be a cube!]

12. Use any method you like to find the point on the ellipsoid with equation $9x^2 + 36y^2 + 4z^2 = 36$ that is closest to the plane with equation $2x + 3y + z = 24$. [<u>There is a simple solution to this</u>. Try drawing a picture first.]