Concourse 18.02 Problem Set 5 – Fall 2018

due Friday, October 12

1. [13.7/20] Given the relation $x^3 + y^3 + z^3 = xyz$ and assuming that $z = z(x, y)$ is implicitly defined by this relation, calculate the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ (as functions of *x*, *y*, and *z*).

2. [13.7/31] Write an equation for the tangent plane to the surface $x^3 + y^3 + z^3 = 5xyz$ at the point $P(2,1,1)$.

3. Two surfaces, S_1 and S_2, are described by the equations: S_1: $xy - x^2 + z^2 = 1$ S_2: $3y = 2xz + y^3$
These surfaces intersect in a curve *C* that contains the point $(1,1,1)$.

   a) Find equations of the tangent planes to S_1 and S_2 respectively at the point $(1,1,1)$.

   b) Find parametric equations for the line tangent to *C* at $(1,1,1)$.

   c) On surface S_1, its equation implicitly defines the variable *x* as a function of the other two variables. Give expressions for the two partial derivatives of this function and evaluate these expressions at the point $(1,1,1)$.

4. [13.7/40 (expanded)] Let $w = f(x, y)$ where the Cartesian coordinates (x, y) are related to the polar coordinates (r,θ) by the equations $x = r\cos\theta$ and $y = r\sin\theta$. Do the following:

   (a) Find expressions for $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$ involving (r,θ), and the partial derivatives $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$.

   (b) Show that $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$.

   [You may find it simpler to start with the right-hand side of the equation.]

5. [13.8/17] Find the directional derivative of the function $f(x, y, z) = \sqrt{xyz}$ at the point $P(2,-1,-2)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k}$.

6. [13.8/44] Suppose that the temperature at the point (x, y, z) in space, with distance measured in kilometers, is given by $w = f(x, y, z) = 10 + xy + xz + yz$ (in degrees Celsius). Find the rate of change (in degrees Celsius per kilometer) of temperature at the point $P(1, 2, 3)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k}$.

7. [13.8/55] A plane tangent to the surface $xyz = 1$ at a point in the first octant cuts off a pyramid from the first octant. Show that any two such pyramids have the same volume.

8. [13.10/7] Find and classify the critical points of the function $f(x, y) = x^3 + y^3 + 3xy + 3$.

9. [13.5/28] Find the maximum and minimum values of the function $f(x, y) = xy^2$ on the plane region *R* that is the circular disk $x^2 + y^2 \le 3$.

10. Use the **Method of Least Squares** to find an equation for the line that best fits the data points $(-3,3)$, $(-2,3)$, $(0,2)$, $(2,1)$, and $(4,-1)$.

11. [13.5/59] A very long rectangle of sheet metal has width L and is to be folded to make a rain gutter (see figure to right). Maximize its volume by maximizing the cross-sectional area shown in the figure.